Weierstrass semigroups on double covers
of plane curves of degree 7

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Weierstrass semigroups on double covers of plane curves of degree 7

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Abstract

We investigate Weierstrass semigroups of ramification points on double covers of plane curves of degree 7. We treat the cases where the Weierstrass semigroups are generated by at most 5 elements and the ramification point is on a total flex.

Keywords: Numerical semigroup, Weierstrass semigroup, Plane curve, Double cover of a curve

1 Introduction

Let \( \mathbb{N}_0 \) be the additive monoid of non-negative integers. A submonoid \( H \) of \( \mathbb{N}_0 \) is called a numerical semigroup if the complement \( \mathbb{N}_0 \setminus H \) is a finite set. The cardinality of \( \mathbb{N}_0 \setminus H \) is said to be the genus of \( H \), which is denoted by \( g(H) \). Let \( C \) be a curve, which means a complete non-singular irreducible algebraic curve over an algebraically closed field \( k \) of characteristic 0 in this article. For a pointed curve \( (C, P) \) of genus \( g \) we set

\[
H(P) = \{ h \in \mathbb{N}_0 \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = hP \}
\]

where \((f)_\infty\) is the polar divisor of the function \( f \). Then \( H(P) \) becomes a numerical semigroup of genus \( g \). We call \( H(P) \) the Weierstrass semigroup of \( P \). A numerical semigroup \( H \) is said to be Weierstrass if there exists a pointed curve \( (C, P) \) such that \( H = H(P) \). For any numerical semigroup \( H \) we define

\[
d_2(H) = \{ h \in \mathbb{N}_0 \mid 2h \in H \},
\]

that is to say, \( d_2(H) \) is the quotient of \( H \) by 2. Then \( d_2(H) \) is also a numerical semigroup. If \( \pi : \tilde{C} \to C \) is a double cover of a curve with a ramification point \( \tilde{P} \) over \( P \), then we have \( d_2(H(\tilde{P})) = H(P) \). Such a numerical semigroup \( H = H(\tilde{P}) \) is said to be of double covering type.

Let \( C \) be a smooth plane curve of degree \( d \geq 2 \) and \( P \) its total flex, i.e., \( \text{ord}_P C.T_P = d \) where \( T_P \) is the tangent line at \( P \) on \( C \) and \( \text{ord}_P C.T_P \) is the multiplicity at \( P \) of the intersection divisor \( C.T_P \) of \( C \) with \( T_P \). Then we have \( H(P) = \langle d - 1, d \rangle \) where for any positive integers \( a_1, \ldots, a_n \) we denote by \( \langle a_1, \ldots, a_n \rangle \) the additive monoid generated by \( a_1, \ldots, a_n \). Conversely, if \((C, P)\) is a pointed curve with \( H(P) = \langle d - 1, d \rangle \), \( d \geq 3 \), then \( C \) is a plane curve with total flex \( P \). In this article we are interested in the double covers \( \pi : \tilde{C} \to C \) of curves with ramification points on the points whose Weierstrass semigroups are \( \langle d - 1, d \rangle \). We pose the following problem:

**TF Hurwitz Problem.** Let \( d \) be a positive integer with \( d \geq 3 \). Let \( H \) be any numerical semigroup with \( d_2(H) = \langle d - 1, d \rangle \) with \( g(H) \geq \frac{3(d - 2)(d - 1)}{2} \). Then is \( H \) of double covering type?

In the above problem TF means total flexes. Under the assumption \( g(H) \geq \frac{3(d - 2)(d - 1)}{2} \) we can construct a double cover \( \pi : \tilde{C} \to C \) with a ramification point \( \tilde{P} \) over \( P \) for a pointed plane curve \((\tilde{C}, \tilde{P})\) with \( H(\tilde{P}) = \langle d - 1, d \rangle \). But
we cannot prove that \( H(P) = H \). TF Hurwitz Problem was solved for \( d \leq 6 \). For \( d = 3 \) the result is classical (for example, see Theorem 3.5 in 1)). If \( d = 4 \), this problem was solved^2. In the cases \( d = 5 \) and 6 the problem are proved in 3)and 4) respectively. We treat the case \( d = 7 \) in this article. Let \( n \) be the minimum odd integer in \( H \). Then we have \( g(H) = (d-1)(d-2) + \frac{n-1}{2} - r \) with a non-negative integer \( r \)^5.

**Main Theorem.** Let \( H \) be a numerical semigroup with \( d_2(H) = (6, 7) \) and \( g(H) \geq 45 \).

i) If \( H \) is generated by 5 elements and \( r \leq 6 \), then it is of double covering type.

ii) If \( H \) is generated by 4 elements and \( H \neq 2\langle 6, 7 \rangle + \langle n, n + 8 \rangle \), then it is of double covering type.

## 2 The classification of \( H \) with \( r \leq 6 \) generated by at most 5 elements

A numerical semigroup \( H \) is said to be an \( m \)-semigroup if \( m \) is the minimum positive integer in \( H \). In this case \( m \) is called the multiplicity of \( H \), which is denoted by \( m(H) \). For an \( m \)-semigroup \( H \) we set

\[
\delta = \min\{ h \in H \mid h \equiv i \bmod m \}
\]

for \( i = 1, \ldots, m - 1 \). The set \( \{m, \delta, \ldots, \delta_{m-1}\} \) is denoted by \( S(H) \), which is called the standard basis for \( H \).

From now on, let \( H \) be a numerical semigroup with \( d_2(H) = H_7 \) where we set \( H_7 = (6, 7) \). We set

\[
n = \min\{ h \in H \mid h \text{ is odd} \}.
\]

We assume \( n \geq 35 \). Then we have \( 2\langle 6, 7 \rangle + n\mathbb{N}_0 \subseteq H \). We note that

\[
S(2\langle 6, 7 \rangle + n\mathbb{N}_0) = \{12, 14, 28, 42, 56, 70\} \cup \{n, n+14, n+28, n+42, n+56, n+70\}.
\]

We associate to \( H \) the diagram where \( \odot \), \( \circ \) and \( \times \) indicate an integer which is in \( M(H) \), \( H \setminus M(H) \) and \( \mathbb{N}_0 \setminus H \) respectively. Here \( M(H) \) denotes the minimal set of generators for the monoid \( H \). Let \( r = r(H) \) be the number of \( \odot \) and \( \circ \). Then \( 0 \leq r \leq 15 \). Moreover, we obtain \( g(H) = 30 + \frac{n-1}{2} - r \). Let \( t(H) \) be the cardinality of the set \( \{u \in M(H) \mid u \text{ is an odd integer distinct from } n\} \).

For example, we associate the following diagram with the numerical semigroup \( H = 2\langle 6, 7 \rangle + \langle n, n + 16, n + 32 \rangle \):

\[
\begin{array}{cccccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\oplus & \times & \times & \times & \times & \\
(n) & \bullet & \odot & \times & \times & \times & \downarrow +12 \\
\downarrow +14 & (n+14) & \bullet & \circ & \odot & \times \\
& (n+28) & \bullet & \circ & \circ & \\
& (n+42) & \bullet & \circ & \\
& (n+56) & \bullet & \\
& (n+70) &
\end{array}
\]

In this case we have \( r = r(H) = 6 \), \( g(H) = 30 + \frac{n-1}{2} - 6 \) and \( t(H) = 2 \).

From here we list numerical semigroups \( H \) with \( d_2(H) = \langle 6, 7 \rangle \), \( r = r(H) \leq 6 \) and \( t(H) = 1 \) or 2. We consider the numerical semigroups with diagrams such that there are no \( \odot \)'s in a left column of the column with \( \circ \) in the diagram below.

(1) Consider

\[
\begin{array}{cccccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \odot \\
(n) & \bullet & \times & \times & \times & \circ & \downarrow +12 \\
\downarrow +14 & (n+14) & \bullet & \times & \times & \circ \\
& (n+28) & \bullet & \times & \circ \\
& (n+42) & \bullet & \circ \\
& (n+56) & \bullet \\
& (n+70) &
\end{array}
\]
Then we have $H = 2H_7 + \langle n, n + 2t_1 \rangle$ with $t_1 = 35 - 6l$ where $l$ is a positive integer with $l \leq 5$.

(2) Consider

\[
\begin{array}{cccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \times \\
(n) & \bullet & \times & \times & \circ & \circ \downarrow +12 \\
\backslash_{n+14} & (n+14) & \bullet & \times & \circ & \circ \\
\end{array}
\]

Then we have $H = 2H_7 + \langle n, n + 2t_1 \rangle$ with $t_1 = 28 - 6l$ where $l$ is a positive integer with $l \leq 3$, $H = 2H_7 + \langle n, n + 2(28 - 12), n + 2t_2 \rangle$ where $t_2 = 35 - 6l$ with $l = 3, 4$ and $H = 2H_7 + \langle n, n + 2(28 - 6), n + 2t_2 \rangle$ where $t_2 = 35 - 6l$ with $2 \leq l \leq 5$.

(3) Consider

\[
\begin{array}{cccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \times \\
(n) & \bullet & \times & \times & \circ & \circ \downarrow +12 \\
\backslash_{n+14} & (n+14) & \bullet & \times & \circ & \circ \\
\end{array}
\]

Then we have $H = 2H_7 + \langle n, n + 2t_1 \rangle$ with $t_1 = 21 - 6l$ where $l$ is a positive integer with $l \leq 2$, $H = 2H_7 + \langle n, n + 2(21 - 6), n + 2(28 - 12) \rangle$ and $H = 2H_7 + \langle n, n + 2(21 - 6), n + 2t_2 \rangle$ where $t_2 = 35 - 6l$ with $l = 2, 3$.

(4) Consider

\[
\begin{array}{cccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \circ & \times & \times & \times & \times \\
(n) & \bullet & \circ & \times & \times & \circ \downarrow +12 \\
\backslash_{n+14} & (n+14) & \bullet & \circ & \times & \circ \\
\end{array}
\]

Then we have $H = 2H_7 + \langle n, n + 2(14 - 6) \rangle$, $H = 2H_7 + \langle n, n + 2(14 - 6), n + 2(28 - 12) \rangle$ and $H = 2H_7 + \langle n, n + 2(14 - 6), n + 2t_2 \rangle$ where $t_2 = 35 - 6l$ with $l = 2, 3$.

(5) Consider

\[
\begin{array}{cccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \circ & \circ & \times & \times & \times \\
(n) & \bullet & \circ & \times & \times & \circ \downarrow +12 \\
\backslash_{n+14} & (n+14) & \bullet & \circ & \times & \circ \\
\end{array}
\]

Then we have $H = 2H_7 + \langle n, n + 2(7 - 6) \rangle$, and $H = 2H_7 + \langle n, n + 2(7 - 6), n + 2(35 - 12) \rangle$. 

Weierstrass semigroups on double covers of plane curves of degree 7 (Kim・米田)
3 The case where $H$ with $r \leq 6$ is generated by 5 elements

By Theorem 2.5 in 3) we know that the following numerical semigroups $H$ with $d_2(H) = (6, 7)$ are of double covering type.

**Theorem 3.1** Let $n$ be an odd number with $n \geq 35$. Let $H$ be a numerical semigroup with $d_2(H) = H_7 = (6, 7)$ which is one of the following type:

(i) $2H_7 + \langle n, n + 2(35 - 12), n + 2t_2 \rangle$ with $t_2 = 7(7 - m) - 6$ where $m$ is an integer with $3 \leq m \leq 6$ and $n \geq (7 - 1)(7 - 2) + 1 + 2m$.

(ii) $2H_7 + \langle n, n + 2(35 - 6l), n + 2(28 - 6l) \rangle$ where $l$ is an integer with $3 \leq l \leq 5$ and $n \geq (7 - 1)(7 - 2) + 3 + 2l$.

(iii) $2H_7 + \langle n, n + 2(21 - 6), n + 2(35 - 18) \rangle$ with $n \geq (7 - 1)(7 - 2) + 11$.

(iv) $2H_7 + \langle n, n + 2(28 - 12), n + 2(35 - 18) \rangle$ with $n \geq (7 - 1)(7 - 2) + 11$.

(v) $2H_7 + \langle n, n + 2(21 - 6), n + 2(28 - 12) \rangle$ with $n \geq (7 - 1)(7 - 2) + 11$.

Then $H$ is of double covering type.

In this section we consider the case where $t(H) = 2$, i.e., $H$ is generated by 5 elements.

**The case (2) in section 2.** By Theorem 3.1 (i), (ii) and (iv), any $H$ with $t(H) = 2$ except

$$H = 2H_7 + \langle n, n + 2(28 - 12), n + 2(35 - 24) \rangle$$

is of double covering type.

**The case (3) in section 2.** By Theorem 3.1 (i), (iii) and (v), any $H$ with $t(H) = 2$ except

$$H = 2H_7 + \langle n, n + 2(21 - 6), n + 2(35 - 24) \rangle$$

is of double covering type.

**The case (4) in section 2.** By Theorem 3.1, $H$ with $t(H) = 2$ which is neither

$$2H_7 + \langle n, n + 2(14 - 6), n + 2(28 - 12) \rangle \text{ nor } 2H_7 + \langle n, n + 2(14 - 6), n + 2(35 - 18) \rangle$$

is of double covering type.

**The case (5) in section 2.** By Theorem 3.1 (i), $H = 2H_7 + \langle n, n + 2(7 - 6), n + 2(35 - 12) \rangle$ is of double covering type.

**Theorem 3.2** Let $C$ be a non-singular plane curve of degree $d \geq 4$. Let $E$ be an effective divisor of degree $d - 1$ on $C$. We set $E = Q_1 + \cdots + Q_{d-1}$ where $Q_i$‘s are points of $C$. Then we have $h^0(E) = 2$ if and only if $Q_1, \ldots, Q_{d-1}$ lie on a line $^{6)}$.

**Theorem 3.3** Let $(C, P)$ be a pointed non-singular plane curve of degree 7 and $H$ a numerical semigroup with $d_2(H) = H(P)$ and $g(H) \geq 45$. Set

$$n = \min \{h \in H \mid h \text{ is odd} \}.$$

We note that

$$g(H) = 30 + \frac{n - 1}{2} - r$$

with some non-negative integer $r$. Let $Q_1, \ldots, Q_r$ be points of $C$ different from $P$ with $h^0(Q_1 + \cdots + Q_r) = 1$. Moreover, assume that $H$ has an expression

$$H = 2d_2(H) + \langle n, n + 2l_1, \ldots, n + 2l_s \rangle$$

with positive integers $l_1, \ldots, l_s$ such that for any curve $C_4$ of degree 4 the inequality $C_4.C \geq (l_i - 1)P + Q_1 + \cdots + Q_r$ implies that $C_4.C \geq l_i P + Q_1 + \cdots + Q_r$, i.e.,

$$h^0(K - (l_i - 1)P - Q_1 - \cdots - Q_r) = h^0(K - l_i P - Q_1 - \cdots - Q_r)$$

where $K$ is a canonical divisor on $C$. Then there is a double cover $\pi : \tilde{C} \to C$ with a ramification point $\tilde{P}$ over $P$ satisfying $H(\tilde{P}) = H$, i.e., $H$ is of double covering type.
Hence, we get

$$D = \frac{n+1}{2} P - (Q_1 + \cdots + Q_r).$$

By the assumption $g(H) \geq 45$ we have

$$\deg(2D - P) = n - 2r = 2g - 59 \geq 90 - 59 = 31 = 2g(C) + 1$$

where $g(C)$ is the genus of the plane curve $C$ of degree 7. Hence, the complete linear system $|2D - P|$ is base-point free. By Theorem 3.2 in [5) we can construct a double cover

$$\pi : \tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \mathcal{O}(-D)) \to C$$

with a ramification point $\tilde{P}$ over $P$ with $H(\tilde{P}) = H$. □

Hereafter, let $C$ be a non-singular plane curve of degree 7 with a total flex $P$ and $Q_1, \ldots, Q_r$ be points of $C$ distinct from $P$. We set $E_r = Q_1 + \cdots + Q_r$.

**Theorem 3.4** $H = 2H_7 + (n, n + 2(28 - 12), n + 2(35 - 24))$ is of double covering type.

**Proof.** In this case $r = 6$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line $L_1$ with $Q_5 \notin L_1$ and $Q_6 \notin L_1$. By Theorem 3.2 we obtain $h^0(Q_1 + \cdots + Q_6) = 1$. Let $C_4$ be a curve of degree 4 with $C_4 \geq 10P + E_6$. Since $C.T^2_\mathbb{P}L_1 \geq 14P + Q_1 + Q_2 + Q_3 + Q_4$, by Bézout’s Theorem (see Theorem p.172 in [7)) we must have $C_4 = T^2_\mathbb{P}L_1L_2$ with a line $L_2$, which implies that $C_4 \geq 14P + E_6$. Thus, we get $h^0(K - 10P - E_6) = h^0(K - 11P - E_6) = h^0(K - 14P - E_6) = 1$. Moreover, we have $h^0(K - 15P - E_6) = 0$. By Theorem 3.3 $H$ is of double covering type. □

**Theorem 3.5** $H = 2H_7 + (n, n + 2(21 - 12), n + 2(35 - 24))$ is of double covering type.

**Proof.** In this case $r = 6$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line $L_1$ with $Q_5 \notin L_1$ and $Q_6 \notin L_1$. Let us take a line $L_P$ which is distinct from $T_P$. Let $Q_5$ and $Q_6$ be on the line $L_P$. Let $C_4$ be a curve of degree 4 with $C_4 \geq 10P + E_6$. We obtain $C_4 = T^2_\mathbb{P}L_1L_2$. Hence, we have

$$h^0(K - 10P - E_6) = h^0(K - 11P - E_6) = h^0(K - 14P - E_6) = h^0(K - 15P - E_6) = 1.$$

By Theorem 3.3 $H$ is of double covering type.

**Theorem 3.6** $H = 2H_7 + (n, n + 2(14 - 6), n + 2(35 - 18))$ is of double covering type.

**Proof.** In this case $r = 6$. Let $L_P$ be a line through $P$ which is distinct from $T_P$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line $L_P$. Let $Q_5$ and $Q_6$ be points such that the line $L_0$ through the two points does not contain $P$. Let $C_4$ be a curve of degree 4 with $C_4 \geq 7P + E_6$. Then we have $C_4 = T_PL_PC_2$ where $C_2$ is a conic containing $Q_5$ and $Q_6$. Hence we get $h^0(K - 7P - E_6) = h^0(K - 8P - E_6)$. Moreover, let $C_4'$ be a curve of degree 4 with $C_4' \geq 16P + E_6$. Then we should have $C_4' = T^2_\mathbb{P}L_PL_0$, which implies that $\text{ord}_P(C_4', C) = 15$. This is a contradiction. Hence, we get $h^0(K - 16P - E_6) = 0$. Thus, $H$ is of double covering type. □

**Theorem 3.7** $H = 2H_7 + (n, n + 2(14 - 6), n + 2(28 - 12))$ is of double covering type.

**Proof.** In this case $r = 6$. Let $L_P$ and $L_P'$ be distinct lines through $P$ different from $T_P$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line $L_P$. Let us take $Q_5$ and $Q_6$ such that the two points lie on the line $L_P'$. Let $C_4$ be a curve of degree 4 with $C_4 \geq 7P + E_6$. Then we have $C_4 = T_PL_PC_2$ where $C_2$ is a conic containing $Q_5$ and $Q_6$. Hence we get $h^0(K - 7P - E_6) = h^0(K - 8P - E_6)$. Moreover, let $C_4'$ be a curve of degree 4 with $C_4' \geq 15P + E_6$. Then we should have $C_4' = T^2_\mathbb{P}L_PL_P'$, which implies that $h^0(K - 15P - E_6) = h^0(K - 16P - E_6) = 1$. Thus, $H$ is of double covering type. □
4 The case where $H$ is generated by 4 elements

In this section we treat the numerical semigroups $H$ with $d_2(H) = (6, 7)$ and $t(H) = 1$. By Theorem 2.5 in 3) we know that the following numerical semigroups $H$ with $d_2(H) = (6, 7)$ are of double covering type.

**Theorem 4.1** Let $n$ be an odd number with $n \geq 35$. Let $H$ be a numerical semigroup which is one of the following:

(i) $2H_7 + \langle n, n + 2t_1 \rangle$ with $t_1 = 35 - l(7 - 1)$ where $l$ is a positive integer with $l \leq 5$ and $n \geq (7 - 1)(7 - 2) + 1 + 2l$.

(ii) $2H_7 + \langle n, n + 2t_1 \rangle$ with $t_1 = 7m - (7 - 1)$ where $m$ is an integer with $3 \leq m \leq 6$ and $n \geq (7 - 1)(7 - 2) - 1 + 2m$.

(iii) $2H_7 + \langle n, n + 2t_1 \rangle$ with $t_1 = 7m - 2(7 - 1)$ where $m$ is an integer with $3 \leq m \leq 5$ and $n \geq (7 - 1)(7 - 2) - 3 + 4m$.

Then $H$ is of double covering type.

With Theorem 4.1, we cannot say that the following three semigroups $H$ with $d_2(H) = (6, 7)$ and $t(H) = 1$ are of double covering type or not.

(1) $2H_7 + \langle n, n + 20 \rangle$  (2) $2H_7 + \langle n, n + 8 \rangle$  (3) $2H_7 + \langle n, n + 6 \rangle$.

To prove that the numerical semigroups in (1) and (3) are of double covering type we need the following:

**Theorem 4.2** (Cayley-Bacharach) (For example, see p. 671 in 7)) Let $C$ be a non-singular plane curve. Let $X_1$ and $X_2$ be two plane curves of degree $d$ and $e$ respectively, meeting in a collection $\Gamma$ of $de$ points of $C$ with multiplicity. Let $Y$ be a curve of degree $d + e - 3$ such that the intersection $Y.C$ contains all but one point of $\Gamma$. Then $Y.C$ contains that remaining point also.

For the case (1) we use the following curve:

**Lemma 4.3** The plane curve of degree 7 defined by the equation

$$(yz^2 - x^3)\left(\frac{1}{2}z^4 + ax^4\right) + (yz^2 + x^3 - 2y^3)\left(\frac{1}{2}z^4 + by^4\right) = 0$$

is nonsingular for general $a$ and $b$.

**Proof.** We have

$$(yz^2 - x^3)\left(\frac{1}{2}z^4 + ax^4\right) + (yz^2 + x^3 - 2y^3)\left(\frac{1}{2}z^4 + by^4\right) = z^4(yz^2 - y^3) + ax^4(yz^2 - x^3) + by^4(yz^2 + x^3 - 2y^3) = F.$$

We will calculate the base locus, i.e., the intersection of the three curves

$$z^4(yz^2 - y^3) = 0, x^4(yz^2 - x^3) = 0 \text{ and } y^4(yz^2 + x^3 - 2y^3) = 0.$$

If $z = 0$, then we have $x = 0$ and $y = 0$. This is a contradiction. Hence, we may set $z = 1$. Thus, we consider the intersection of the following three curves

$$y - y^3 = 0, x^4(y - x^3) = 0 \text{ and } y^4(y + x^3 - 2y^3) = 0.$$

Hence, we have $y = 0, 1 \text{ or } -1$. Let $y = 0$. Then we have $x = 0$. Hence, we obtain the point $(0 : 0 : 1)$. Let $y = 1$. Then $x = 1, \omega \text{ or } \omega^2$ where $\omega$ is a primitive cubic root of unity. Hence, we get the three points $(1 : 1 : 1), (\omega : 1 : 1)$ and $(\omega^2 : 1 : 1)$. Let $y = -1$. Then $x = -1, -\omega \text{ or } -\omega^2$. Thus, we get the three points $(-1 : -1 : 1), (-\omega : -1 : 1)$ and $(-\omega^2 : -1 : 1)$. Therefore, the base locus consists of the seven points. The partial differentials of $F$ are the following:

$$F_x = 4ax^3(yz^2 - x^3) - 3ax^6 + 3bx^2y^4 = ax^3(4yz^2 - 7x^3) + 3bx^2y^4$$

$$F_y = z^6 - 3yz^2z^4 + ax^4z^2 + 4by^4(yz^2 + x^3 - 2y^3) + by^4(z^2 - 6y^2) = z^6 - 3yz^2z^4 + ax^4z^2 + by^4(5yz^2 + 4x^3 - 14y^2)$$

and $F_z = 4z^3(yz^2 - y^3) + 2ax^4yz + 2by^5z$. 
For general \( a \) and \( b \) we have
\[
F_x(0,0,1) = 1 \neq 0, F_x(1,1,1) = \omega^3 = 0, F_x(\omega, 1 : 1) = \omega^{10} = 0, F_x(\omega^2, 1 : 1) = -3a + 3b \neq 0.
\]
\[
F_x(-1, -1, 1) = 3a + b \neq 0, F_x(-\omega, -1, 1) = -3a + 3b \neq 0 \text{ and } F_x(-\omega^2, -1, 1) = -3a + 3b \neq 0.
\]
Hence, the plane curve defined by \( F = 0 \) is non-singular for general \( a \) and \( b \) by Bertini’s theorem (for example, see p.137 in 7)).

\[ \square \]

**Theorem 4.4** Let \( n \) be an odd number with \( n \geq 43 \). Then \( 2H_7 + \langle n, n + 20 \rangle \) is of double covering type.

**Proof.** In this case \( r = 6 \). Let \( C \) be the non-singular plane curve of degree 7 in Lemma 4.3. We set \( P = (0 : 0 : 1) \).
Then we have \( C.T_P = 7P \), in this case \( T_P \) is the line defined by \( y = 0 \). Let \( C_{13} \) and \( C_{32} \) be the cubics defined by the equations \( yz^2 = x^3 = 0 \) and \( yz^2 + x^3 - 2y^3 = 0 \), respectively. Then the intersection \( C_{31}.C_{32} \) of \( C_{31} \) and \( C_{32} \) is \( 3P + \sum_{i=1}^{6} Q_i \) where \( Q_1 = (1 : 1 : 1), Q_2 = (1 : 1 : \omega), Q_3 = (1 : 1 : \omega^2), Q_4 = (1 : -1 : -1), Q_5 = (1 : -1 : -\omega) \) and \( Q_6 = (1 : -1 : -\omega^2) \). Since the six points \( Q_1, \ldots, Q_6 \) are not on a line. Hence by Theorem 3.2 we get \( n^3(Q_1, \ldots, Q_6) = 1 \). Let \( C_4 \) be a curve of degree 4 with \( C_4.C \geq 9P + E_6 \). Then we obtain \( C_3.C \geq 3P + \sum_{i=1}^{6} Q_i \). Hence we get \( C_4.C \geq 10P + E_6 \). By Theorem 3.3 the numerical semigroup \( 2H_7 + \langle n, n + 20 \rangle \) is of double covering type.

\[ \square \]

**Lemma 4.5** The plane curve of degree 7 defined by the equation
\[
(yz^2 - x^3) \left( \frac{1}{2} z^4 + ax^4 \right) + (yz^3 + x^3 z - 2y^4) \left( \frac{1}{2} z^3 + by^3 \right) = 0
\]
is nonsingular for general \( a \) and \( b \).

**Proof.** We have
\[
(yz^2 - x^3) \left( \frac{1}{2} z^4 + ax^4 \right) + (yz^3 + x^3 z - 2y^4) \left( \frac{1}{2} z^3 + by^3 \right) = yz^6 - y^4(z^3 + ax^3) + by^3(yz^3 + x^3 z - 2y^4) = F.
\]
The base locus is the intersection of
\[
z^3(yz^3 - y^4) = 0, x^4(yz^2 - x^3) = 0 \text{ and } y^3(yz^3 + x^3 z - 2y^4) = 0.
\]
If \( z = 0 \), then we have \( x = 0 \) and \( y = 0 \). This is a contradiction. Hence, we may set \( z = 1 \). Thus, we consider the intersection of the following three curves
\[
y - y^4 = 0, x^4(y - x^3) = 0 \text{ and } y^3(y + x^3 - 2y^4) = 0.
\]
Since we have \( y - y^4 = y(1 - y^3) = 0 \), we obtain \( y = 0, y = 1, y = \omega \) or \( y = \omega^2 \). If \( y = 0 \), then \( x = 0 \). Hence, we get the point \( (0 : 0 : 1) \). If \( y = 1 \), then \( x^3 = 1 \). Thus, we have the three points \((1 : 1 : 1), (\omega : 1 : 1) \) and \((\omega^2 : 1 : 1) \). If \( y = \omega \), then we obtain the three points \((\zeta : \omega : 1), (\zeta^4 : \omega, 1) \) and \((\zeta^7 : \omega : 1) \) where \( \zeta \) is a primitive 9-th root of unity. If \( y = \omega^2 \), then we obtain the three points \((\zeta^2 : \omega^2 : 1), (\zeta^5 : \omega^2 : 1) \) and \((\zeta^8 : \omega^2 : 1) \). The partial differentials of \( F \) are the following:
\[
F_x = ax^3(4yz^2 - 7x^3) + 3bx^2y^3 z, F_y = z^6 - 4y^3z^3 + ax^4z^2 + by^2(3x^3 z + 4yz^3 - 14y^4)
\]
and \( F_z = 6y^5z - 3y^4z^2 + 2ax^4yz + by^3(3yz^2 + x^3) \).

Hence, we have
\[
F_y(0,0,1) = 1 \neq 0, F_x(1,1,1) = -3a + 3b \neq 0
\]
for general \( a \) and \( b \). For the remaining eight points the values of the function \( F_z \) are not zero for general \( a \) and \( b \). Hence the plane curve of degree 7 is non-singular.

\[ \square \]
Theorem 4.6  Let \( n \) be an odd number with \( n \geq 49 \). Then \( 2H_7 + \langle n, n + 6 \rangle \) is of double covering type.

Proof. In this case we have \( r = 9 \). Let \( C \) be the non-singular plane curve of degree 7 in Lemma 4.5. We set \( P = (0 : 0 : 1) \). Then we have \( C.T_P = 7P \), in this case \( T_P \) is the line defined by \( y = 0 \). Let \( C_{31} \) be the cubic defined by the equation \( yz^2 - x^3 = 0 \) and \( C_{41} \) be the quartic defined by the equation \( yz^3 + x^3z - 2y^4 = 0 \). We may assume that \( z = 1 \). Hence we consider the equations \( y - x^3 = 0 \) and \( y + x^3 - 2y^4 = 0 \), which imply that \( x^3(y^3 - 1) = 0 \). Let \( \eta \) be a primitive 9-th root of unity. We set \( Q_l = \langle \eta^l : \eta^{3l} : 1 \rangle \) for \( l = 0, 1, 2, \ldots, 8 \). Then the intersection divisor \( C_{31}.C_{41} = 3P + \sum_{l=0}^{8} Q_l \).

Let \( C_4 \) be a curve of degree 4 with \( C_4.C \geq 2P + E_9 \). Then by Theorem 4.2 we get \( C_3.C \geq 3P + \sum_{i=0}^{8} Q_i \). We want to show that \( h^0(K - E_9) = 6 \). Let \( C_4 \) be a curve of degree 4 with \( C_4.C \geq E_9 \), i.e., it is defined by the equation

\[
F_4(x, y, z) = c_{400}x^4 + c_{410}x^3y + c_{301}x^3z + c_{220}x^2y^2 + c_{211}x^2yz + c_{202}x^2z^2 + c_{130}xy^3 + c_{121}xy^2z + c_{103}xz^3 + c_{040}y^4 + c_{031}y^3z + c_{022}y^2z^2 + c_{013}yz^3 + c_{004}z^4 = 0
\]

satisfying \( F_4(\eta^l, \eta^{3l}, 1) = 0 \) for \( l = 0, 1, \ldots, 8 \). The rank of the matrix of the coefficients of the system of linear equations \( F_4(\eta^l, \eta^{3l}, 1) = 0 \) \((l = 0, 1, \ldots, 8)\) with 15 variables \( c_{ijk}, i + j + k = 4 \) is 9, because some 9 by 9 minor of the matrix is Vandermonde’s determinant. Hence, we get \( h^0(K - E_9) = 15 - 9 = 6 \), which implies that \( h^0(E_9) = 1 \). Thus, \( 2H_7 + \langle n, n + 6 \rangle \) is of double covering type.

\( \square \)

We do not know whether the remaining numerical semigroup \( 2H_7 + \langle n, n + 8 \rangle \) is of double covering type or not.

Acknowledgment. This work was supported by JSPS KAKENHI Grant Number18K03228.

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神奈川工科大学研究報告
B・44 理工学編 通巻 44 号

令和 2 年 3 月 1 日 発行
編集兼発行者 神奈川工科大学
〒 243-0292 神奈川県厚木市下荻野1030
電話 046-241-6221
印刷者 株式会社スクールパートナーズ

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