On 7-semigroups of genus 9 generated by 5 elements

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Abstract

This paper is devoted to constructing an affine toric variety from a non-primitive 7-semigroup of genus 9 generated by 5 elements such that the monomial curve associated with the 7-semigroup is a specialization of the affine toric variety.

Key Words: Affine toric variety, 7-semigroup, Numerical semigroup of genus 9

§1. Introduction.

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ is a finite set. When H is a numerical semigroup, the cardinality of the set $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. For a positive integer n a numerical semigroup H is said to be an *n-semigroup* if the minimum positive integer in H is n. Let C be a complete non-singular irreducible algebraic curve of genus g over an algebraically closed field k of characteristic 0. For any point P of C, $H(P)$ denotes the set of integers which are pole orders at P of regular functions on $C \backslash \{P\}$. Then $H(P)$ is a numerical semigroup of genus g. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. It is known that any numerical semigroup of genus $g \leqq 8$ is Weierstrass[1),2)]. An n-semigroup H is said to be *primitive* if the largest integer not in H is less than $2n$. We already showed that every primitive numerical semigroup of genus 9 is Weierstrass[3)]. Moreover, we know that for any $n \leqq 5$ every n-semigroup is Weierstrass[4),5),6)]. We are interested in non-primitive n-semigroups of genus 9 for $n \geqq 6$. By the way there is only one non-primitive n-semigroup of genus 9 with $n \geqq 8$. The unique semigroup H_8 is generated by $8, 10, 11, 12, 13, 14$ and 15. It was proved that H_8 is Weierstrass[7)]. We want to study on all non-primitive 7-semigroups of genus 9 generated by 5 elements, which are the following 4 semigroups :

$$< 7, 9, 10, 11, 12 > , \ < 7, 9, 10, 11, 13 > , \ < 7, 8, 11, 12, 13 > \text{ and } < 7, 9, 10, 12, 13 >$$

where for any positive integers $b_1, b_2, \ldots, b_n$ the semigroup generated by $b_1, b_2, \ldots, b_n$ is denoted by $< b_1, b_2, \ldots, b_n >$. In this paper from each H in the above 4 semigroups we construct an affine toric variety such that the monomial curve associated with H is a specialization of the affine toric variety, because we know that if we can construct such an affine toric variety from a given numerical semigroup, then the numerical semigroup is Weierstrass[5)]. We treat the semigroups $< 7, 9, 10, 11, 12 >$, $< 7, 9, 10, 11, 13 >$, $< 7, 8, 11, 12, 13 >$ and $< 7, 9, 10, 12, 13 >$ in Sections 2, 3, 4 and 5 respectively.

§2. On the 7- semigroup generated by $7, 9, 10, 11$ and 12.

Let H be the 7-semigroup generated by $a_1 = 7, a_2 = 9, a_3 = 10, a_4 = 11$ and $a_5 = 12$. We set

$$\alpha_i = \min\{\alpha \in \mathbb{Z}_{>0} \mid \alpha a_i \in < a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_5 >\}$$

for $i = 1, 2, \ldots, 5$. Then we get

$$\alpha_1 = 3 \text{ and } \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 2.$$

In fact, we have

$$3a_1 = a_2 + a_5,\ 2a_2 = a_1 + a_4,\ 2a_3 = a_2 + a_4,\ 2a_4 = a_3 + a_5 \text{ and } 2a_5 = 2a_1 + a_3.$$

We want to determine the relations

$$\sum_{i=1}^{5} \mu_i a_i = \sum_{i=1}^{5} \nu_i a_i \text{ with } \mu_i \nu_i = 0,\ 0 \leqq \mu_i < \alpha_i \text{ and } 0 \leqq \nu_i < \alpha_i \text{ for all } i.$$

The following table on the sum of two elements is used:

	$a_1 = 7$	$2a_1 = 14$	$a_2 = 9$	$a_3 = 10$	$a_4 = 11$	$a_5 = 12$
$a_1 = 7$	–	–	16	17	18	19
$2a_1 = 14$	–	–	23	24	25	26
$a_2 = 9$	–	–	–	19	20	21
$a_3 = 10$	–	–	–	–	21	22
$a_4 = 11$	–	–	–	–	–	23
$a_5 = 12$	–	–	–	–	–	–

Considering the above table and the equalities

$$a_1 + a_2 + a_3 + a_4 + a_5 = 49 \text{ and } 2a_1 + a_2 + a_3 + a_4 + a_5 = 56$$

we get the desired relations

$$a_1 + a_5 = a_2 + a_3,\ a_2 + a_5 = a_3 + a_4 \text{ and } 2a_1 + a_2 = a_4 + a_5.$$

Let $\varphi_H : k[X_1, \ldots, X_5] \longrightarrow k[H] = k[t^h]_{h \in H}$ be the k-algebra homomorphism sending X_i to t^{a_i}. We want to show that the ideal $I_H = \operatorname{Ker} \varphi_H$ is generated by the polynomials associated with the above eight relations where for example we associate the polynomial $X_1^2 X_2 - X_4 X_5$ with the relation $2a_1 + a_2 = a_4 + a_5$. Let J be the ideal generated by the polynomials associated with the above eight relations. It suffices to check that J contains the polynomial f associated with the relation

$$\sum_{i=1}^{5} \mu_i a_i = \sum_{i=1}^{5} \nu_i a_i \text{ with } \mu_i \nu_i = 0,\ 0 \leqq \mu_i \text{ and } 0 \leqq \nu_i \text{ for all } i.$$

Considering the generators of J we may assume that $\mu_i \geqq \alpha_i$ for some i.

Let $i = 1$. Then we may assume that $f = X_1^{\mu_1} X_2^{\mu_2} X_5^{\mu_5} - X_3^{\nu_3} X_4^{\nu_4} \in I_H$, because the relation $3a_1 = a_2 + a_5$ belongs to the set R of the eight relations. Moreover, the relations $2a_3 = a_2 + a_4$, $2a_4 = a_3 + a_5$ and $a_3 + a_4 = a_2 + a_5$ are contained in the set R. Hence, we may decrease the degree of f.

Let $i = 2$. Then we may assume that $f = X_2^{\mu_2} X_1^{\mu_1} X_4^{\mu_4} - X_3^{\nu_3} X_5^{\nu_5} \in I_H$. The relations $2a_3 = a_2 + a_4$, $2a_5 = 2a_1 + a_3$ and $2a_4 = a_3 + a_5$ are contained in the set R. Hence, we may decrease the degree of f.

Let $i = 3$. Then we may assume that $f = X_3^{\mu_3} X_2^{\mu_2} X_4^{\mu_4} - X_1^{\nu_1} X_5^{\nu_5} \in I_H$. If $\nu_1 \geqq \alpha_1 = 3$, using $3a_1 = a_2 + a_5$ we may decrease the degree of f. If $\nu_1 = 1$ or 2, then $\nu_5 \geqq 1$. Using $a_1 + a_5 = a_2 + a_3$

we may decrease the degree of f. Hence, we may assume that $\nu_1 = 0$ and $\nu_5 \geqq \alpha_5 = 2$. Using $2a_5 = 2a_1 + a_3$ we may decrease the degree of f.

Let $i = 4$. Then we may assume that $f = X_4^{\mu_4}X_3^{\mu_3}X_5^{\mu_5} - X_1^{\nu_1}X_2^{\nu_2} \in I_H$. If $\nu_1 \geqq \alpha_1 = 3$, using $3a_1 = a_2 + a_5$ we may decrease the degree of f. If $\nu_2 \geqq \alpha_2 = 2$, using $2a_2 = a_1 + a_4$ we may decrease the degree of f. Hence, we have $\nu_2 = 1$. If $\nu_1 = 1$, this contradicts $\mu_4 \geqq 2$. If $\nu_1 = 2$, using $2a_1 + a_2 = a_4 + a_5$ we may decrease the degree of f.

Let $i = 5$. Then we may assume that $f = X_5^{\mu_5}X_1^{\mu_1}X_3^{\mu_3} - X_2^{\nu_2}X_4^{\nu_4} \in I_H$. If $\nu_2 \geqq 2$ or $\nu_4 \geqq 2$, using $2a_2 = a_1 + a_4$ and $2a_4 = a_3 + a_5$ we may decrease the degree of f. Hence, we may assume that $\nu_2 = \nu_4 = 1$, which contradicts $\mu_5 \geqq \alpha_5 = 2$. Hence, we get Ker $\varphi_H = J$.

When we set

$$\alpha_{12} = \alpha_{15} = \alpha_{21} = \alpha_{24} = \alpha_{32} = \alpha_{34} = \alpha_{43} = \alpha_{45} = 1,\ \alpha_{51} = 2 \text{ and } \alpha_{53} = 1,$$

the above eight relations, which form a system of generators for the ideal Ker φ_H, are written in the following way:

$$(\alpha_{21} + \alpha_{51})a_1 = \alpha_{12}a_2 + \alpha_{15}a_5 \tag{1}$$
$$(\alpha_{12} + \alpha_{32})a_2 = \alpha_{21}a_1 + \alpha_{24}a_4 \tag{2}$$
$$(\alpha_{43} + \alpha_{53})a_3 = \alpha_{32}a_2 + \alpha_{34}a_4 \tag{3}$$
$$(\alpha_{24} + \alpha_{34})a_4 = \alpha_{43}a_3 + \alpha_{45}a_5 \tag{4}$$
$$(\alpha_{15} + \alpha_{45})a_5 = \alpha_{51}a_1 + \alpha_{53}a_3 \tag{5}$$
$$\alpha_{21}a_1 + \alpha_{45}a_5 = \alpha_{12}a_2 + \alpha_{53}a_3 \tag{6}$$
$$\alpha_{32}a_2 + \alpha_{45}a_5 = \alpha_{53}a_3 + \alpha_{24}a_4 \tag{7}$$
$$\alpha_{51}a_1 + \alpha_{32}a_2 = \alpha_{24}a_4 + \alpha_{15}a_5. \tag{8}$$

When ${}^t(n)$ means the equation whose right side (resp. left side) is the left side (resp. right side) of the equation (n), we get

$$(5) = {}^t(1) + {}^t(2) + {}^t(3) + {}^t(4),\ (6) = {}^t(2) + {}^t(3) + {}^t(4),\ (7) = {}^t(3) + {}^t(4) \text{ and } (8) = (1) + (2).$$

Hence, the relations (1), (2), (3) and (4) form a fundamental system of relations between a_1, a_2, a_3, a_4 and a_5. Using the relations (1), (2), (3) and (4) we associate the vectors $\mathbf{b}_i$'s in $\mathbb{Z}^6$ ($i = 1, \ldots, 10$) with $\alpha_{21}a_1$, $\alpha_{51}a_1$, $\alpha_{12}a_2$, $\alpha_{32}a_2$, $\alpha_{43}a_3$, $\alpha_{53}a_3$, $\alpha_{15}a_5$, $\alpha_{24}a_4$, $\alpha_{34}a_4$ and $\alpha_{45}a_5$ respectively, where $\mathbf{b}_i = \mathbf{e}_i$, i.e., the unit vector in $\mathbb{Z}^6$ whose i-th component is 1 and the other components are 0, $\mathbf{b}_7 = (1, 1, -1, 0, 0, 0)$, $\mathbf{b}_8 = (-1, 0, 1, 1, 0, 0)$, $\mathbf{b}_9 = (0, 0, 0, -1, 1, 1)$ and $\mathbf{b}_{10} = (-1, 0, 1, 0, 0, 1)$. Let S be the subsemigroup of $\mathbb{Z}^6$ generated by $\mathbf{b}_1, \ldots, \mathbf{b}_{10}$. We want to show that S is *saturated*, i.e., $nr \in S$ with $n \in \mathbb{N}$ and $r \in \mathbb{Z}^6$ implies that $r \in S$. To prove that S is saturated it suffices to show that $\sum_{i=1}^{10} \mathbb{R}_+\mathbf{b}_i \cap \mathbb{Z}^6 \subseteqq S$ where $\mathbb{R}_+$ is the set of non-negative real numbers. Let us take $\mathbf{b} \in \sum_{i=1}^{10} \mathbb{R}_+\mathbf{b}_i \cap \mathbb{Z}^6$. We may assume that $\mathbf{b} = \sum_{i=1}^{10} \lambda_i \mathbf{b}_i$ with $0 \leqq \lambda_i < 1$ for all i. If we set $\mathbf{b} = (\mu_1, \ldots, \mu_6)$, then we get

$$\mu_1 = \lambda_1 + \lambda_7 - \lambda_8 - \lambda_{10},\ \mu_2 = \lambda_2 + \lambda_7,\ \mu_3 = \lambda_3 - \lambda_7 + \lambda_8 + \lambda_{10},$$

$$\mu_4 = \lambda_4 + \lambda_8 - \lambda_9,\ \mu_5 = \lambda_5 + \lambda_9 \text{ and } \mu_6 = \lambda_6 + \lambda_9 + \lambda_{10}.$$

Since $\mathbf{b} \in \mathbb{Z}^6$, we get $\mu_1 = -1$ or 0 or 1 and $\mu_i \geqq 0$ for all $i \geqq 2$. If $\mu_1 = 0$ or 1, then $\mathbf{b} \in S$. If $\mu_1 = -1$, then we must have $\mu_3 \geqq 1$ and $\mu_6 \geqq 1$. Hence, we may assume that $\mathbf{b} = (-1, 0, 1, 0, 0, 1) = \mathbf{b}_{10} \in S$. Therefore, S is saturated. Using the above result we may construct an affine toric variety such that the monomial curve Spec $k[H]$ associated with H is a specialization of the affine toric variety. In fact, let

$\eta : k[Y_1, \ldots, Y_{10}] \longrightarrow k[S][X_1, \ldots, X_5]$ be the k-algebra homomorphism sending Y_i to $T^{\mathbf{b}_i} - g_i$ for all i where $k[S] = k[T^s]_{s \in S}$ is the semigroup algebra over k associated with S and g_i's ($i = 1, \ldots, 10$) in $k[X_1, \ldots, X_5]$ are the monomials derived from $\alpha_{21}a_1$, $\alpha_{51}a_1$, $\alpha_{12}a_2$, $\alpha_{32}a_2$, $\alpha_{43}a_3$, $\alpha_{53}a_3$, $\alpha_{15}a_5$, $\alpha_{24}a_4$, $\alpha_{34}a_4$ and $\alpha_{45}a_5$ respectively. Then we get a fiber product as follows:

$$\begin{array}{ccc} \text{Spec } k[H] & \rightarrow & \text{Spec } k[S][X_1, \ldots, X_5] \\ \downarrow & \square & \downarrow^{a}\eta \\ \text{Spec } k & \xrightarrow{\rho} & \text{Spec } k[Y_1, \ldots, Y_{10}] \end{array}$$

where ρ sends the unique point to the origin of Spec $k[Y_1, \ldots, Y_{10}]$. Here, Spec $k[S][X_1, \ldots, X_5]$ is an affine toric variety, because S is saturated. Using the above fiber product we can construct a pointed curve (C, P) such that $H(P) = H$ (See Corollary 4.9 in 5)).

§3. On the 7- semigroup generated by $7, 9, 10, 11$ and 13.

Let H be the 7-semigroup generated by $a_1 = 7, a_2 = 9, a_3 = 10, a_4 = 11$ and $a_5 = 13$. Then we get

$$\alpha_1 = 3 \text{ and } \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 2,$$

where α_i's are as in Section 2. In fact, we have

$$3a_1 = a_3 + a_4,\ 2a_2 = a_1 + a_4,\ 2a_3 = a_1 + a_5,\ 2a_4 = a_2 + a_5 \text{ and } 2a_5 = a_1 + a_2 + a_3.$$

To determine the remaining relations between a_1, a_2, a_3, a_4 and a_5 the following table is used:

	$a_1 = 7$	$2a_1 = 14$	$a_2 = 9$	$a_3 = 10$	$a_4 = 11$	$a_5 = 13$
$a_1 = 7$	–	–	16	17	18	20
$2a_1 = 14$	–	–	23	24	25	27
$a_2 = 9$	–	–	–	19	20	22
$a_3 = 10$	–	–	–	–	21	23
$a_4 = 11$	–	–	–	–	–	24
$a_5 = 13$	–	–	–	–	–	–

Considering the equalities

$$a_1 + a_2 + a_3 + a_4 + a_5 = 50 \text{ and } 2a_1 + a_2 + a_3 + a_4 + a_5 = 57$$

we get the desired relations

$$a_1 + a_5 = a_2 + a_4,\ 2a_1 + a_2 = a_3 + a_5 \text{ and } 2a_1 + a_3 = a_4 + a_5.$$

Let φ_H, I_H and J be as in Section 2. To prove that $I_H = J$ we use the method as in the previous section.

Let $i = 1$. Then we may assume that $f = X_1^{\mu_1}X_3^{\mu_3}X_4^{\mu_4} - X_2^{\nu_2}X_5^{\nu_5} \in I_H$ with $\mu_3\mu_4 \neq 0$.. Using the relations $2a_2 = a_1 + a_4$ and $2a_5 = a_1 + a_2 + a_3$ we may decrease the degree of f.

Let $i = 2$. Then we may assume that $f = X_2^{\mu_2}X_1^{\mu_1}X_4^{\mu_4} - X_3^{\nu_3}X_5^{\nu_5} \in I_H$ with $\mu_1\mu_4 \neq 0$. We may decrease the degree of f, because of $2a_3 = a_1 + a_5$ and $2a_5 = a_1 + a_2 + a_3$.

Let $i = 3$. Then we may assume that $f = X_3^{\mu_3}X_1^{\mu_1}X_5^{\mu_5} - X_2^{\nu_2}X_4^{\nu_4} \in I_H$ with $\mu_1\mu_5 \neq 0$. We may decrease the degree of f, because of $2a_2 = a_1 + a_4$ and $2a_4 = a_2 + a_5$.

Let $i = 4$. Then we may assume that $f = X_4^{\mu_4}X_2^{\mu_2}X_5^{\mu_5} - X_1^{\nu_1}X_3^{\nu_3} \in I_H$ with $\mu_4 \geqq \alpha_4$ or $\mu_2\mu_5 \neq 0$. We may decrease the degree of f, because of $3a_1 = a_3 + a_4$ and $2a_3 = a_1 + a_5$.

Let $i = 5$. By the relation $2a_5 = a_1 + a_2 + a_3$ we may decrease the degree of f. Hence, we get Ker $\varphi_H = J$.

When we set

$$\alpha_{13} = \alpha_{14} = \alpha_{21} = \alpha_{24} = \alpha_{31} = \alpha_{35} = \alpha_{42} = \alpha_{45} = \alpha_{51} = \alpha_{52} = \alpha_{53} = 1,$$

the above eight relations, which form a system of generators for the ideal Ker φ_H, are written in the following way:

$$(\alpha_{21} + \alpha_{31} + \alpha_{51})a_1 = \alpha_{13}a_3 + \alpha_{14}a_4 \tag{9}$$

$$(\alpha_{42} + \alpha_{52})a_2 = \alpha_{21}a_1 + \alpha_{24}a_4 \tag{10}$$

$$(\alpha_{13} + \alpha_{53})a_3 = \alpha_{31}a_1 + \alpha_{35}a_5 \tag{11}$$

$$(\alpha_{14} + \alpha_{24})a_4 = \alpha_{42}a_2 + \alpha_{45}a_5 \tag{12}$$

$$(\alpha_{35} + \alpha_{45})a_5 = \alpha_{51}a_1 + \alpha_{52}a_2 + \alpha_{53}a_3 \tag{13}$$

$$\alpha_{21}a_1 + \alpha_{45}a_5 = \alpha_{52}a_2 + \alpha_{14}a_4 \tag{14}$$

$$(\alpha_{31} + \alpha_{51})a_1 + \alpha_{52}a_2 = \alpha_{13}a_3 + \alpha_{45}a_5 \tag{15}$$

$$(\alpha_{21} + \alpha_{51})a_1 + \alpha_{53}a_3 = \alpha_{14}a_4 + \alpha_{35}a_5. \tag{16}$$

Hence we get

$$(13) = {}^t(9) + {}^t(10) + {}^t(11) + {}^t(12),\ (14) = {}^t(10) + {}^t(12),\ (15) = (9) + (10) + (12) \text{ and } (16) = (9) + (11).$$

Thus, the relations (9), (10), (11) and (12) form a fundamental system of relations between a_1, a_2, a_3, a_4 and a_5. Using the relations (9), (10), (11) and (12) we associate the vectors $\mathbf{b}_i$'s in $\mathbb{Z}^7$ $(i = 1, \ldots, 11)$ with $\alpha_{21}a_1$, $\alpha_{31}a_1$,$\alpha_{51}a_1$, $\alpha_{13}a_3$, $\alpha_{42}a_2$, $\alpha_{52}a_2$, $\alpha_{53}a_3$, $\alpha_{14}a_4$, $\alpha_{24}a_4$, $\alpha_{35}a_5$ and $\alpha_{45}a_5$ respectively, where $\mathbf{b}_i = \mathbf{e}_i$, i.e., the i-th unit vector in $\mathbb{Z}^7$, $\mathbf{b}_8 = (1, 1, 1, -1, 0, 0, 0)$, $\mathbf{b}_9 = (-1, 0, 0, 0, 1, 1, 0)$, $\mathbf{b}_{10} = (0, -1, 0, 1, 0, 0, 1)$ and $\mathbf{b}_{11} = (0, 1, 1, -1, 0, 1, 0)$. Let S be the subsemigroup of $\mathbb{Z}^7$ generated by $\mathbf{b}_1, \ldots, \mathbf{b}_{11}$. We will show that S is saturated. Let us take $\mathbf{b} \in \sum_{i=1}^{11} \mathbb{R}_+\mathbf{b}_i \cap \mathbb{Z}^7$. We may assume that $\mathbf{b} = \sum_{i=1}^{11} \lambda_i \mathbf{b}_i$ with $0 \leqq \lambda_i < 1$ for all i. If we set $\mathbf{b} = (\mu_1, \ldots, \mu_7)$, then we get

$$\mu_1 = \lambda_1 + \lambda_8 - \lambda_9,\ \mu_2 = \lambda_2 + \lambda_8 - \lambda_{10} + \lambda_{11},\ \mu_3 = \lambda_3 + \lambda_8 + \lambda_{11},$$

$$\mu_4 = \lambda_4 - \lambda_8 + \lambda_{10} - \lambda_{11},\ \mu_5 = \lambda_5 + \lambda_9,\ \mu_6 = \lambda_6 + \lambda_9 + \lambda_{11} \text{ and } \mu_7 = \lambda_7 + \lambda_{10}.$$

If $\mu_4 = -1$, then

$$\mu_1 \geqq 0,\ \mu_2 \geqq 1,\ \mu_3 \geqq 1,\ \mu_5 \geqq 0,\ \mu_6 \geqq 1 \text{ and } \mu_7 \geqq 0.$$

Hence, we may assume that $\mathbf{b} = \mathbf{b}_{11} \in S$. Thus, S is saturated. In the same way as in Section 2 we may construct an affine toric variety such that the monomial curve Spec $k[H]$ is a specialization of the affine toric variety.

§4. On the 7- semigroup generated by 7, 8, 11, 12 and 13.

Let H be the 7-semigroup generated by $a_1 = 7, a_2 = 8, a_3 = 11, a_4 = 12$ and $a_5 = 13$. Then we get

$$\alpha_1 = \alpha_2 = 3 \text{ and } \alpha_3 = \alpha_4 = \alpha_5 = 2,$$

where α_i's are as in Section 2. In fact, we have

$$3a_1 = a_2 + a_5 \tag{17}$$

$$3a_2 = a_3 + a_5 \tag{18}$$

$$2a_3 = 2a_1 + a_2 \tag{19}$$

$$2a_4 = a_3 + a_5 \tag{20}$$

$$2a_5 = 2a_1 + a_4 \tag{21}$$

We have the following table:

	$a_1=7$	$2a_1=14$	$a_2=8$	$2a_2=16$	$a_3=11$	$a_4=12$	$a_5=13$
$a_1=7$	–	–	15	23	18	19	20
$2a_1=14$	–	–	22	30	25	26	27
$a_2=8$	–	–	–	–	19	20	21
$2a_2=16$	–	–	–	–	27	28	29
$a_3=11$	–	–	–	–	–	23	24
$a_4=12$	–	–	–	–	–	–	25
$a_5=13$	–	–	–	–	–	–	–

Considering the above table and the equalities

$$a_1+a_2+a_3+a_4+a_5=51,\ 2a_1+a_2+a_3+a_4+a_5=58=2\times 29,$$

$$a_1+2a_2+a_3+a_4+a_5=59 \text{ and } 2a_1+2a_2+a_3+a_4+a_5=66=2\times 33$$

we get the relations

$$a_1+a_4=a_2+a_3 \tag{22}$$

$$a_1+a_5=a_2+a_4 \tag{23}$$

$$2a_1+a_3=a_4+a_5 \tag{24}$$

$$2a_1+a_5=2a_2+a_3 \tag{25}$$

$$a_1+2a_2=a_3+a_4 \tag{26}$$

Let φ_H, I_H and J be as in Section 2. To prove that $I_H=J$ we use the method as in the previous section.

Let $i=1$. Then we may assume that $f=X_1^{\mu_1}X_2^{\mu_2}X_5^{\mu_5}-X_3^{\nu_3}X_4^{\nu_4}\in I_H$. Using the relations (19), (20) and (26) we may decrease the degree of f.

Let $i=2$. Then we may assume that $f=X_2^{\mu_2}X_3^{\mu_3}X_5^{\mu_5}-X_1^{\nu_1}X_4^{\nu_4}\in I_H$. Using the relations (17), (20) and (22) we may decrease the degree of f.

Let $i=3$. Then we may assume that $f=X_3^{\mu_3}X_1^{\mu_1}X_2^{\mu_2}-X_4^{\nu_4}X_5^{\nu_5}\in I_H$. Using the relations (20), (21) and (24) we may decrease the degree of f.

Let $i=4$. Then we may assume that $f=X_4^{\mu_4}X_3^{\mu_3}X_5^{\mu_5}-X_1^{\nu_1}X_2^{\nu_2}\in I_H$. Using the relations (17), (18), (19) and (26) we may decrease the degree of f.

Let $i=5$. Then we may assume that $f=X_5^{\mu_5}X_1^{\mu_1}X_4^{\mu_4}-X_2^{\nu_2}X_3^{\nu_3}\in I_H$. Using the relations (18), (19) and (22) we may decrease the degree of f.

When we set

$$\beta_1=1,\ \beta_1'=2 \text{ and } \beta_2=\beta_2'=\beta_3=\beta_3'=\beta_4=\beta_5=\beta_5'=1,$$

the above ten relations, which form a system of generators for the ideal Ker φ_H, are written in the following way:

$$(25)\ 2\beta_1a_1+\beta_5a_5=2\beta_2'a_2+\beta_3a_3$$

$$(19)\ (\beta_3+\beta_3')a_3=2\beta_1a_1+\beta_2a_2$$

$$(23)\ \beta_1a_1+\beta_5a_5=\beta_2'a_2+\beta_4a_4$$

$$(24)\ \beta_1'a_1+\beta_3a_3=\beta_4a_4+\beta_5'a_5$$

$$(17)\ (\beta_1+\beta_1')a_1=\beta_2'a_2+\beta_5'a_5$$

$$(18)\ (\beta_2+2\beta_2')a_2=\beta_3'a_3+\beta_5a_5$$

$$(20)\ 2\beta_4a_4=\beta_3a_3+\beta_5a_5$$

$$(21)\ (\beta_5+\beta_5')a_5=\beta_1'a_1+\beta_4a_4$$

$$(22)\ \beta_1a_1+\beta_4a_4=\beta_2'a_2+\beta_3a_3$$

$$(26)\ \beta_1a_1+(\beta_2+\beta_2')a_2=\beta_3'a_3+\beta_4a_4.$$

Since we get

$$(17)={}^t(23)+(24)+(25),\ (18)={}^t(19)+{}^t(25),\ (20)=2\times{}^t(23)+(25),$$

$$(21)=2\times(23)+{}^t(24)+{}^t(25),\ (22)={}^t(23)+(25),\ (26)={}^t(19)+(23)+{}^t(25),$$

(25), (19), (23) and (24) form a fundamental system for the relation module. Using the relations (25), (19), (23) and (24) we associate the vectors $\mathbf{b}_i$'s in $\mathbb{Z}^5$ $(i=1,\dots,9)$ with β_1a_1, β_5a_5, $\beta_2'a_2$, $\beta_3'a_3$, $\beta_1'a_1$, β_3a_3, β_2a_2,β_4a_4 and $\beta_5'a_5$ respectively, where $\mathbf{b}_i=\mathbf{e}_i$, i.e., the i-th unit vector in $\mathbb{Z}^5$, $\mathbf{b}_6=(2,1,-2,0,0)$, $\mathbf{b}_7=(0,1,-2,1,0)$, $\mathbf{b}_8=(1,1,-1,0,0)$ and $\mathbf{b}_9=(1,0,-1,0,1)$. Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1,\dots,\mathbf{b}_9$. We will show that S is saturated. Let us take $\mathbf{b}\in\sum_{i=1}^{9}\mathbb{R}_+\mathbf{b}_i\cap\mathbb{Z}^5$. We may assume that $\mathbf{b}=\sum_{i=1}^{9}\lambda_i\mathbf{b}_i$ with $0\leqq\lambda_i<1$ for all i. If we set $\mathbf{b}=(\mu_1,\dots,\mu_5)$, then we get

$$\mu_1=\lambda_1+2\lambda_6+\lambda_8+\lambda_9,\ \mu_2=\lambda_2+\lambda_6+\lambda_7+\lambda_8,\ \mu_3=\lambda_3-2\lambda_6-2\lambda_7-\lambda_8-\lambda_9,$$

$$\mu_4=\lambda_4+\lambda_7 \text{ and } \mu_5=\lambda_5+\lambda_9.$$

Hence, we obtain $\mu_3\geqq-5$.

Let $\mu_3=-5$, i.e., $2\lambda_6+2\lambda_7+\lambda_8+\lambda_9=\lambda_3+5$. Then

$$\mu_1=\lambda_1+\lambda_3+5-2\lambda_7\geqq4,\ \mu_2=\lambda_2+\lambda_3+5-\lambda_6-\lambda_7-\lambda_9\geqq3,\ \mu_4=\lambda_4+\lambda_7=1 \text{ and } \mu_5=\lambda_5+\lambda_9=1.$$

Hence, we may assume that $\mathbf{b}=(4,3,-5,1,1)=\mathbf{b}_6+\mathbf{b}_7+\mathbf{b}_8+\mathbf{b}_1+\mathbf{b}_5\in S$.

Let $\mu_3=-4$, i.e., $2\lambda_6+2\lambda_7+\lambda_8+\lambda_9=\lambda_3+4$. Then $\lambda_7\neq0$, which implies that $\mu_4=1$. Moreover, $\mu_1\geqq3$ and $\mu_2\geqq2$. Hence, we may assume that $\mathbf{b}=(3,2,-4,1,0)=\mathbf{b}_6+\mathbf{b}_7+\mathbf{b}_1\in S$.

Let $\mu_3=-3$, i.e., $2\lambda_6+2\lambda_7+\lambda_8+\lambda_9=\lambda_3+3$. If $\lambda_7=0$, then $\lambda_9\neq0$, which implies that $\mu_5=1$. Moreover, $\mu_1=\lambda_1+\lambda_3+3$ and $\mu_2=\lambda_2+\lambda_3+3-\lambda_6-\lambda_9$. Hence, we may assume that $\mathbf{b}=(3,2,-3,0,1)=\mathbf{b}_6+\mathbf{b}_9+\mathbf{b}_2\in S$. If $\lambda_7\neq0$ and $\lambda_9\neq0$, then we may assune that $\mathbf{b}=(2,1,-3,1,1)=\mathbf{b}_7+\mathbf{b}_9+\mathbf{b}_1\in S$. If $\lambda_7\neq0$ and $\lambda_9=0$, then we may assune that $\mathbf{b}=(2,2,-3,1,0)=\mathbf{b}_7+\mathbf{b}_8+\mathbf{b}_1\in S$.

Let $\mu_3=-2$, i.e., $2\lambda_6+2\lambda_7+\lambda_8+\lambda_9=\lambda_3+2$. Then $\mu_2=\lambda_2+\lambda_3+2-\lambda_6-\lambda_7-\lambda_9\geqq1$. In fact, if $\mu_2=0$, then we get

$$-2=\mu_3=\lambda_3-2\lambda_6-2\lambda_7-\lambda_8-\lambda_9=\lambda_3-2(\lambda_6+\lambda_7+\lambda_9)-\lambda_8+\lambda_9=\lambda_3-2(\lambda_2+\lambda_3+2)-\lambda_8+\lambda_9\leqq-3,$$

which is a contradiction. If $\lambda_7=0$ and $\lambda_9=0$, then we may assume that $\mathbf{b}=(2,2,-2,0,0)=2\mathbf{b}_8\in S$. If $\lambda_7=0$ and $\lambda_9\neq0$, then we may assume that $\mathbf{b}=(2,1,-2,0,1)=\mathbf{b}_8+\mathbf{b}_9\in S$. If $\lambda_7\neq0$, then we may assume that $\mathbf{b}=(1,1,-2,1,0)=\mathbf{b}_7+\mathbf{b}_1\in S$.

Let $\mu_3=-1$, i.e., $2\lambda_6+2\lambda_7+\lambda_8+\lambda_9=\lambda_3+1$. We have $\mu_2=\lambda_2+\lambda_6+\lambda_7+\lambda_8\geqq1$, because one of λ_6, λ_7 and λ_9 is non-zero. If $\lambda_7\neq0$, then we may assume that $\mathbf{b}=(0,1,-1,1,0)=\mathbf{b}_7+\mathbf{b}_3\in S$. If $\lambda_7=0$, then we may assume that $\mathbf{b}=(1,1,-1,0,0)=\mathbf{b}_8\in S$. Thus, S is saturated. In the same way as in Section 2 we may construct an affine toric variety such that the monomial curve Spec $k[H]$ is a specialization of the affine toric variety.

§5. On the 7- semigroup generated by $7, 9, 10, 12$ and 13.

Let H be the 7-semigroup generated by $a_1 = 7, a_2 = 9, a_3 = 10, a_4 = 12$ and $a_5 = 13$. Then we get

$$\alpha_1 = \alpha_2 = 3 \text{ and } \alpha_3 = \alpha_4 = \alpha_5 = 2,$$

where α_i's are as in Section 2. In fact, we have

$$3a_1 = a_2 + a_4 \tag{27}$$

$$3a_2 = 2a_1 + a_5 \tag{28}$$

$$2a_3 = a_1 + a_5 \tag{29}$$

$$2a_4 = 2a_1 + a_3 \tag{30}$$

$$2a_5 = a_1 + a_2 + a_3 \tag{31}$$

We have the following table:

	$a_1 = 7$	$2a_1 = 14$	$a_2 = 9$	$2a_2 = 18$	$a_3 = 10$	$a_4 = 12$	$a_5 = 13$
$a_1 = 7$	–	–	16	25	17	19	20
$2a_1 = 14$	–	–	23	32	24	26	27
$a_2 = 9$	–	–	–	–	19	21	22
$2a_2 = 18$	–	–	–	–	28	30	31
$a_3 = 10$	–	–	–	–	–	22	23
$a_4 = 12$	–	–	–	–	–	–	25
$a_5 = 13$	–	–	–	–	–	–	–

Considering the above table and the equalities

$$a_1 + a_2 + a_3 + a_4 + a_5 = 51,\ 2a_1 + a_2 + a_3 + a_4 + a_5 = 58 = 2 \times 29,$$

$$a_1 + 2a_2 + a_3 + a_4 + a_5 = 60 = 2 \times 30 \text{ and } 2a_1 + 2a_2 + a_3 + a_4 + a_5 = 67$$

we get the relations

$$a_1 + a_4 = a_2 + a_3 \tag{32}$$

$$a_2 + a_5 = a_3 + a_4 \tag{33}$$

$$2a_1 + a_2 = a_3 + a_5 \tag{34}$$

$$a_1 + 2a_2 = a_4 + a_5 \tag{35}$$

$$2a_2 + a_4 = a_1 + a_3 + a_5 \tag{36}$$

Let φ_H, I_H and J be as in Section 2. To prove that $I_H = J$ we use the method as in the previous section.

Let $i = 1$. Then we may assume that $f = X_1^{\mu_1} X_2^{\mu_2} X_4^{\mu_4} - X_3^{\nu_3} X_5^{\nu_5} \in I_H$. Using the relations (29), (31) and (34) we may decrease the degree of f.

Let $i = 2$. Then we may assume that $f = X_2^{\mu_2} X_1^{\mu_1} X_5^{\mu_5} - X_3^{\nu_3} X_4^{\nu_4} \in I_H$. Using the relations (29), (30) and (33) we may decrease the degree of f.

Let $i = 3$. Then we may assume that $f = X_3^{\mu_3} X_1^{\mu_1} X_5^{\mu_5} - X_2^{\nu_2} X_4^{\nu_4} \in I_H$. Using the relations (28), (30) and (27) we may decrease the degree of f.

Let $i = 4$. Then we may assume $f = X_4^{\mu_4} X_1^{\mu_1} X_3^{\mu_3} - X_2^{\nu_2} X_5^{\nu_5} \in I_H$. Using the relations (28), (31) and (33) we may decrease the degree of f.

Let $i = 5$. We use the relations (30) and (31).

When we set

$$\beta_1 = \beta_1' = \beta_1'' = \beta_2 = \beta_2' = \beta_3 = \beta_3' = \beta_4 = \beta_5 = \beta_5' = 1,$$

the above ten relations, which form a system of generators for the ideal Ker φ_H, are written in the following way:

$$(34)\ (\beta_1+\beta_1')a_1+\beta_2'a_2=\beta_3'a_3+\beta_5a_5$$

$$(35)\ \beta_1a_1+(\beta_2+\beta_2')a_2=\beta_4a_4+\beta_5a_5$$

$$(33)\ \beta_2a_2+\beta_5'a_5=\beta_3a_3+\beta_4a_4$$

$$(30)\ 2\beta_4a_4=(\beta_1+\beta_1'')a_1+\beta_3'a_3$$

$$(27)\ (\beta_1+\beta_1'+\beta_1'')a_1=\beta_2a_2+\beta_4a_4$$

$$(28)\ (2\beta_2+\beta_2')a_2=(\beta_1'+\beta_1'')a_1+\beta_5a_5$$

$$(29)\ (\beta_3+\beta_3')a_3=\beta_1'a_1+\beta_5'a_5$$

$$(31)\ (\beta_5+\beta_5')a_5=\beta_1a_1+\beta_2'a_2+\beta_3a_3$$

$$(32)\ \beta_1'a_1+\beta_4a_4=\beta_2a_2+\beta_3'a_3$$

$$(36)\ (\beta_2+\beta_2')a_2+\beta_4a_4=\beta_1''a_1+\beta_3'a_3+\beta_5a_5$$

Since we get

$$(27)={}^t(30)+(34)+{}^t(35),\ (28)=(30)+{}^t(34)+2\times(35),\ (29)={}^t(33)+{}^t(34)+(35),$$

$$(31)=(33)+{}^t(35),\ (32)=(34)+{}^t(35)\text{ and }(36)=(30)+(35),$$

(34), (35), (33) and (30) form a fundamental system of relations between a_1, a_2, a_3, a_4 and a_5.

Using the relations (34), (35), (33) and (30) we associate the vectors $\mathbf{b}_i$'s in $\mathbb{Z}^6$ $(i=1,\dots,10)$ with β_1a_1, $\beta_1'a_1$, $\beta_2'a_2$, $\beta_3'a_3$, β_2a_2, $\beta_5'a_5$, β_5a_5, β_4a_4, β_3a_3 and $\beta_1''a_1$ respectively, where $\mathbf{b}_i=\mathbf{e}_i$, i.e., the i-th unit vector in $\mathbb{Z}^6$, $\mathbf{b}_7=(1,1,1,-1,0,0)$, $\mathbf{b}_8=(0,-1,0,1,1,0)$, $\mathbf{b}_9=(0,1,0,-1,0,1)$ and $\mathbf{b}_{10}=(-1,-2,0,1,2,0)$. Let S be the subsemigroup of $\mathbb{Z}^6$ generated by $\mathbf{b}_1,\dots,\mathbf{b}_{10}$. We will show that S is saturated. Let us take $\mathbf{b}\in\sum_{i=1}^{10}\mathbb{R}_+\mathbf{b}_i\cap\mathbb{Z}^6$. We may assume that $\mathbf{b}=\sum_{i=1}^{10}\lambda_i\mathbf{b}_i$ with $0\leqq\lambda_i<1$ for all i. If we set $\mathbf{b}=(\mu_1,\dots,\mu_6)$, then we get

$$\mu_1=\lambda_1+\lambda_7-\lambda_{10},\ \mu_2=\lambda_2+\lambda_7-\lambda_8+\lambda_9-2\lambda_{10},\ \mu_3=\lambda_3+\lambda_7,$$

$$\mu_4=\lambda_4-\lambda_7+\lambda_8-\lambda_9+\lambda_{10},\ \mu_5=\lambda_5+\lambda_8+2\lambda_{10}\text{ and }\mu_6=\lambda_6+\lambda_9.$$

Hence, we obtain $\mu_2\geqq-2$ and $\mu_4\geqq-1$. Let $\mu_2=-2$, i.e., $\lambda_2+\lambda_7+\lambda_9+2=\lambda_8+2\lambda_{10}$. Then

$$\mu_4=\lambda_2+\lambda_4+2-\lambda_{10}\geqq2\text{ and }\mu_5=\lambda_5+\lambda_2+\lambda_7+\lambda_9+2\geqq2.$$

Thus, we may assume that $\mathbf{b}=(0,-2,0,2,2,0)=\mathbf{b}_{10}+\mathbf{b}_1+\mathbf{b}_4\in S$. Let $\mu_2=-1$. In the same way as in the case $\mu_2=-2$ we may assume that $\mathbf{b}=(0,-1,0,1,1,0)=\mathbf{b}_8\in S$. Let $\mu_2\geqq0$ and $\mu_4=-1$, i.e., $\lambda_4+\lambda_8+\lambda_{10}+1=\lambda_7+\lambda_9$. Then we get

$$\mu_1=\lambda_1+\lambda_4+\lambda_8+1-\lambda_9\geqq1\text{ and }\mu_2=\lambda_2+\lambda_4-\lambda_{10}+1\geqq1.$$

Moreover, in view of $\lambda_7\neq0$ and $\lambda_9\neq0$ we have $\mu_3=1$ and $\mu_6=1$. Thus, we may assume that $\mathbf{b}=(1,1,1,-1,0,1)=b_7+b_6\in S$. Thus, S is saturated. In the same way as in Section 2 we may construct an affine toric variety such that the monomial curve Spec $k[H]$ is a specialization of the affine toric variety.

Combining the results in Sections 2, 3, 4 and 5 with Corollary 4.9 in 5) we get the following main theorem:

Theorem. *Every non-primitive* 7*-semigroup of genus* 9 *generated by* 5 *elements is Weierstrass.*

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