

# Weierstrass points on a non-singular plane curve of degree 7

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## Abstract

We determine all the candidates for Weierstrass semigroups of points on a non-singular plane curve of degree 7.

Key Words: Plane curve, Weierstrass semigroup of a point

## §1. Introduction.

Let  $C$  be a complete non-singular irreducible algebraic curve of genus  $g$  over an algebraically closed field  $k$  of characteristic 0, which is called a *non-singular curve* in this paper. For any point  $P$  of  $C$ ,  $H(P)$  denotes the set of non-negative integers which are pole orders at  $P$  of regular functions on  $C \setminus \{P\}$ . Then  $H(P)$  is a subsemigroup of the additive semigroup  $\mathbb{N}_0$  of non-negative integers whose complement  $G(P) = \mathbb{N}_0 \setminus H(P)$  consists of  $g$  elements. In this paper we are interested in the semigroup  $H(P)$  where  $P$  is a point on a non-singular plane curve of degree 7. Why do we study on non-singular plane curves of degree 7? The reason is that we have the following results on non-singular plane curves of degree  $\leq 6$ : Let  $P$  be a point on a non-singular plane curve  $C_0$  of degree 1 or 2. Since the genus of the curve  $C_0$  is 0, we have  $H(P) = \mathbb{N}_0$ . Any non-singular plane curve  $E$  of degree 3 is elliptic. So we get  $H(P) = \langle 2, 3 \rangle$  for any point  $P$  of  $E$  where for any positive integers  $a_1, a_2, \dots, a_n$  the semigroup generated by these integers is denoted by  $\langle a_1, a_2, \dots, a_n \rangle$ . A non-hyperelliptic curve of genus 3 is realized by a non-singular plane curve of degree 4, and vice-versa. For a point  $P$  of the non-hyperelliptic curve the semigroup  $H(P)$  is one of the semigroups  $\langle 3, 4 \rangle$ ,  $\langle 3, 5, 7 \rangle$  and  $\langle 4, 5, 6, 7 \rangle$ . In the case of a point  $P$  on a non-singular plane curve of degree 5 there are eight candidates for the semigroups  $H(P)$ . We know that any candidate is attained at a point on a non-singular plane curve of degree 5<sup>(1)</sup>. Let  $C_6$  be a non-singular plane curve of degree 6 with point  $P$ . We can determine all the candidates for  $H(P)$ <sup>(1), (2)</sup>. Moreover, these semigroups are showed to be Weierstrass, where a numerical semigroup  $H$  is *Weierstrass* if there is a pointed non-singular curve  $(C, Q)$  such that  $H = H(Q)$ .

In Section 2 we determine all the candidates for Weierstrass semigroups of inflection points on a non-singular plane curve of degree 7. In the case of the Weierstrass semigroup of a non-inflection point on a non-singular plane curve of degree 7 all the candidates for Weierstrass semigroups are given in Section 3.

## §2. The Weierstrass semigroup of an inflection point.

We always assume that  $C$  is a non-singular plane curve of degree 7 and that  $P$  is a point on  $C$ . If  $\Gamma_1$  and  $\Gamma_2$  are two plane curves and  $P \in \Gamma_1 \cap \Gamma_2$  such that no common component of  $\Gamma_1$  and  $\Gamma_2$  contains

$P$ , then  $I(\Gamma_1 \cap \Gamma_2, P)$  denotes the intersection multiplicity of  $\Gamma_1$  and  $\Gamma_2$  at  $P$ . Since plane curves of degree 4 cut out the canonical linear system on  $C$ , we have

$$G(P) = \mathbb{N}_0 \setminus H(P) = \{I(C \cap \gamma, P) + 1 \mid \gamma \text{ is a curve of degree 4 in } \mathbb{P}^2\}.$$

Let  $T$  be the tangent line to  $C$  at  $P$ , let  $L_1$  be a line on  $\mathbb{P}^2$  containing  $P$  but not coinciding with  $T$  and let  $L_0$  be a line on  $\mathbb{P}^2$  not containing  $P$ . Using the sums of multiples of  $T$ ,  $L_1$  and  $L_0$  and intersecting with  $C$  we obtain a subset of the complement  $G(P)$ . We use the following notations: For any non-negative integers  $\mu$  and  $\nu$  with  $\mu + \nu \leq 4$  the curve  $\mu T + \nu L_1 + (4 - \mu - \nu)L_0$  is denoted by  $C(\mu, \nu)$ .

In this section we are devoted to the case where  $P$  is an inflection point on  $C$ , i.e.,  $I(C \cap T, P) \geq 3$ . The following is proved in Coppens-Kato<sup>2)</sup> even if  $I(C \cap T, P) = d$  or  $d-1$  or  $d-2$  where  $C$  is of degree  $d$ . But we give a proof, because we want to show our method of calculation of the gap sequence.

**Proposition 2.1.** (1) If  $I(C \cap T, P) = 7$ , then we have  $H(P) = \langle 6, 7 \rangle$ .

(2) If  $I(C \cap T, P) = 6$ , then we have  $H(P) = \langle 6, 11, 16, 21, 26, 31 \rangle$ .

(3) If  $I(C \cap T, P) = 5$ , then we have  $H(P) = \langle 10, 14, 15, 18, 19, 22, 23, 26, 27, 31 \rangle$ .

*Proof.* (1) Since  $I(C \cap T, P) = 7$ , we have  $I(C \cap C(\mu, \nu), P) = 7\mu + \nu$  for  $0 \leq \mu + \nu \leq 4$ . Hence, the gap sequence at  $P$  is

$$\{7\mu + \nu + 1 \mid 0 \leq \mu, 0 \leq \nu, \mu + \nu \leq 4\} = \{1, 2, 3, 4, 5, 8, 9, 10, 11, 15, 16, 17, 22, 23, 29\},$$

which implies that  $H(P) = \langle 6, 7 \rangle$ .

(2) Since  $I(C \cap T, P) = 6$ , we have  $I(C \cap C(\mu, \nu), P) = 6\mu + \nu$  for  $0 \leq \mu + \nu \leq 4$ . Hence, the gap sequence at  $P$  is

$$\{6\mu + \nu + 1 \mid 0 \leq \mu, 0 \leq \nu, \mu + \nu \leq 4\} = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 19, 20, 25\},$$

which implies that  $H(P) = \langle 6, 11, 16, 21, 26, 31 \rangle$ .

(3) Since  $I(C \cap T, P) = 5$ , we have  $I(C \cap C(\mu, \nu), P) = 5\mu + \nu$  for  $0 \leq \mu + \nu \leq 4$ . Hence, the gap sequence at  $P$  is

$$\{5\mu + \nu + 1 \mid 0 \leq \mu, 0 \leq \nu, \mu + \nu \leq 4\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 16, 17, 21\},$$

which implies that  $H(P) = \langle 10, 14, 15, 18, 19, 22, 23, 26, 27, 31 \rangle$ . □

In the case of  $I(C \cap T, P) \leq 4$  we can not determine the gap sequence at  $P$  only using the above method. The following result is useful for getting the remaining gaps at  $P$ : If  $\Gamma_1$  and  $\Gamma_2$  are two plane curves, we have

$$I(\Gamma_1 \cap \Gamma_2, P) \geq \min\{I(\Gamma_1 \cap C, P), I(\Gamma_2 \cap C, P)\}.$$

From now on, to describe the minimal set of generators for the semigroup  $H(P)$  or the finite set  $G(P)$  we use the following notation: For any integers  $a$  and  $b$  with  $a < b$  the symbol  $a \longrightarrow b$  means  $b - a + 1$  consecutive integers  $a, a + 1, \dots, b - 1, b$ .

**Theorem 2.2.** If  $I(C \cap T, P) = 4$ , then  $H(P)$  is one of the following semigroups:

- i)  $\langle 12, 15, 16, 18 \longrightarrow 23, 25, 26 \rangle$     ii)  $\langle 12, 15, 16, 18 \longrightarrow 23, 25, 29 \rangle$
- iii)  $\langle 12, 15, 16, 18 \longrightarrow 23, 26, 29 \rangle$     iv)  $\langle 12, 15, 16, 18 \longrightarrow 22, 25, 26, 29 \rangle$
- v)  $\langle 12, 15, 16, 18, 19, 20, 21, 23, 25, 26, 29 \rangle$     vi)  $\langle 12, 15, 16, 18, 19, 20, 22, 23, 25, 26, 29 \rangle$
- vii)  $\langle 12, 15, 16, 18, 19, 21, 22, 23, 25, 26, 29 \rangle$     viii)  $\langle 12, 15, 16, 18, 20, 21, 22, 23, 25, 26, 29 \rangle$
- ix)  $\langle 12, 15, 16, 19, 20, 21, 22, 23, 25, 26, 29 \rangle$     x)  $\langle 12, 15, 18 \longrightarrow 23, 25, 26, 28, 29 \rangle$
- xi)  $\langle 12, 16, 18 \longrightarrow 23, 25, 26, 27, 29 \rangle$     xii)  $\langle 15, 16, 18 \longrightarrow 29 \rangle$

*Proof.* Since  $I(C \cap T, P) = 4$ , we have  $I(C \cap C(\mu, \nu), P) = 4\mu + \nu$  for  $0 \leq \mu + \nu \leq 4$ . Hence, the set  $G(P)$  of the gaps at  $P$  contains the set

$$S = \{4\mu + \nu + 1 \mid 0 \leq \mu, 0 \leq \nu, \mu + \nu \leq 4\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17\}.$$

Thus, there is only one element  $\gamma$  in the set  $G(P)$  such that  $\gamma \notin S$ , which implies that  $G(P) = S \cup \{\gamma\}$ .

If  $\Gamma$  is a plane conic curve not containing the tangent line  $T$  at  $P$ , then

$$2 \geq I(\Gamma \cap T, P) \geq \text{Min}\{I(\Gamma \cap C, P), I(T \cap C, P)\}.$$

Since  $I(T \cap C, P) = 4$ , we must have  $2 \geq I(\Gamma \cap C, P)$ . Hence, from the intersection multiplicity of  $C$  and any plane curve of degree 4 containing the conic  $\Gamma$  at  $P$  we cannot get the element  $\gamma$ .

If  $\Gamma$  is a plane cubic curve not containing the tangent line  $T$  at  $P$ , then

$$3 \geq I(\Gamma \cap T, P) \geq \text{Min}\{I(\Gamma \cap C, P), I(T \cap C, P)\}.$$

Since  $I(T \cap C, P) = 4$ , we must have  $3 \geq I(\Gamma \cap C, P)$ . Hence, from the intersection multiplicity of  $C$  and any plane curve of degree 4 containing the cubic  $\Gamma$  at  $P$  we cannot get the element  $\gamma$ .

Let  $\Gamma$  be a plane quartic curve not containing the tangent line  $T$  at  $P$ . Then we have  $I(\Gamma \cap C, P) \leq 28$ . Since  $\mathbb{N}_0 \setminus (S \cup \{\gamma\})$  is an additive semigroup,  $I(\Gamma \cap C, P)$  is distinct from 23, 26 and 27. If  $I(\Gamma \cap C, P) = 28$ , then  $\gamma = 29$ . Hence we have  $S \cup \{\gamma\} = \{1, 2, \dots, 11, 13, 14, 17, 29\}$ , which implies i). If  $I(\Gamma \cap C, P) = 25$ , we get ii). If  $I(\Gamma \cap C, P) = 24, 22, 21, 20, 19, 18, 17, 15, 14, 11$ , we get iii), iv), v), vi), vii), viii), ix), x), xi), xii) respectively.  $\square$

In the case  $I(C \cap T, P) = 3$  the description of  $G(P)$  is simpler than that of  $H(P)$ .

**Theorem 2.3.** *If  $I(C \cap T, P) = 3$ , then  $G(P)$  is one of the following sets:*

- i)  $\{1 \rightarrow 14, \gamma\}$  for some  $\gamma$  with  $15 \leq \gamma \leq 29$ ,
- ii)  $\{1 \rightarrow 13, 15, \gamma\}$  for some  $\gamma$  with  $16 \leq \gamma \leq 29$  and  $\gamma \neq 28$ ,
- iii)  $\{1 \rightarrow 13, \gamma_1, \gamma_2\}$  for some  $\gamma_1$  and  $\gamma_2$  with  $16 \leq \gamma_1 < \gamma_2 \leq 27$ ,
- iv)  $\{1 \rightarrow 11, 13, 14, 16, \gamma\}$  for some  $\gamma$  with  $15 \leq \gamma \leq 28$  and  $\gamma \neq 16, 24, 27$ ,
- v)  $\{1 \rightarrow 11, 13, \gamma + 1, \gamma + 2, \gamma + 4\}$  for some  $\gamma$  with  $13 \leq \gamma \leq 19$  or  $\gamma = 21$ .

*Proof.* Since  $I(C \cap T, P) = 3$ , we have  $I(C \cap C(\mu, \nu), P) = 3\mu + \nu$  for  $0 \leq \mu + \nu \leq 4$ . Hence, the set  $G(P)$  of the gaps at  $P$  contains the set

$$S = \{3\mu + \nu + 1 \mid 0 \leq \mu, 0 \leq \nu, \mu + \nu \leq 4\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}.$$

Thus, we need more three elements to describe the set  $G(P)$ .

If  $\Gamma$  is a plane conic curve not containing the tangent line  $T$  at  $P$ , then

$$2 \geq I(\Gamma \cap T, P) \geq \text{Min}\{I(\Gamma \cap C, P), I(T \cap C, P)\}.$$

Since  $I(T \cap C, P) = 3$ , we must have  $2 \geq I(\Gamma \cap C, P)$ . Hence, from the intersection multiplicity of  $C$  and any plane curve of degree 4 containing the conic  $\Gamma$  at  $P$  we cannot get any element in the set  $G(P)$  not belonging to  $S$ .

Let  $\Gamma$  be a plane cubic curve not containing the tangent line  $T$  at  $P$ . Then we have  $I(\Gamma \cap C, P) \leq 21$ . We don't need to consider the case where  $I(\Gamma \cap C, P) \leq 7$  or  $I(\Gamma \cap C, P) = 9$ . In fact, the dimension of the cubics in  $\mathbb{P}^2$  is of dimension 10. So, there exists a cubic  $\Gamma$  such that  $I(\Gamma \cap C, P) = 8$  or  $I(\Gamma \cap C, P) \geq 10$ . First, we consider the case  $I(\Gamma \cap C, P) = 8$ . Then we have  $I((\Gamma + T) \cap C, P) = 11$ , which implies  $12 \in G(P)$ . If  $M_4$  is a plane quartic curve not containing the tangent line  $T$  at  $P$ , then  $I(M_4 \cap C, P) \leq 28$ . Hence, the set  $G(P)$  must be one of i), ii) and iii). Second, we investigate the case  $I(\Gamma \cap C, P) = 10$ . In this case, i) holds. Third, we consider the case  $I(\Gamma \cap C, P) = 11$ . Then either i)

with  $\gamma = 15$  or ii) holds. Fourth, let  $I(\Gamma \cap C, P) = 12$ . In this case, we get i) with  $\gamma = 16$  or iv). Last, we consider the remaining cases  $13 \leq I(\Gamma \cap C, P) \leq 21$ . If  $I(\Gamma \cap C, P) = 20$ , we have  $24 \in G(P)$ , which is a contradiction. Otherwise,  $G(P)$  is v) with  $\gamma = I(\Gamma \cap C, P)$ .  $\square$

### §3. The Weierstrass semigroup of a non-inflection point.

Let the notation be as in Section 2. Moreover, we assume that  $P$  is a non-inflection point on  $C$ , i.e.,  $I(T \cap C, P) = 2$ . In this section we will find all the candidates for Weierstrass semigroups  $H(P)$  of non-inflection points  $P$ .

We have  $I(C \cap C(\mu, \nu), P) = 2\mu + \nu$  for  $0 \leq \mu + \nu \leq 4$ . Hence, the set  $G(P)$  of the gaps at  $P$  contains the set

$$S = \{2\mu + \nu + 1 \mid 0 \leq \mu, 0 \leq \nu, \mu + \nu \leq 4\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

We set  $i_2 = \max_{\Gamma: \text{conic}} \{I(\Gamma \cap C, P)\} \leq 14$ . We note that  $i_2 \geq 5$ , because the dimension of the conics in  $\mathbb{P}^2$  is of dimension 6.

**Theorem 3.1.** *If  $i_2 \geq 9$ , then we obtain  $G(P) = \{1 \rightarrow 9, i_2 + 1 \rightarrow i_2 + 5, 2i_2 + 1\}$ .*

*Proof.* If  $i_2 \geq 9$ , we get  $i_2 + 1 \geq 10$ . Let  $\Gamma$  be a conic such that  $I(\Gamma \cap C, P) = i_2$ . Since we consider the intersection multiplicity of the curve  $C$  and plane curves  $\Gamma + 2L_0$ ,  $\Gamma + L_0 + L_1$ ,  $\Gamma + 2L_1$ ,  $\Gamma + L_1 + T$ ,  $\Gamma + 2T$ , respectively, at  $P$ , we get the gaps  $i_2 + 1 \rightarrow i_2 + 5$  at  $P$ . Moreover, from  $I(2\Gamma \cap C, P) = 2i_2$  we get the gap  $2i_2 + 1$ .  $\square$

In the case where  $5 \leq i_2 \leq 8$  we give the set  $G(P)$  separately.

**Theorem 3.2.** *If  $i_2 = 8$ , then we have  $G(P) = \{1 \rightarrow 13, 17, \gamma\}$  with  $14 \leq \gamma \leq 16$  or  $18 \leq \gamma \leq 27$ .*

*Proof.* Let  $\Gamma$  be a conic such that  $I(\Gamma \cap C, P) = 8$ . Then the intersection multiplicities of the curve  $C$  and the curves  $\Gamma + L_0 + L_1$ ,  $\Gamma + 2L_1$ ,  $\Gamma + L_1 + T$ ,  $\Gamma + 2T$ ,  $2\Gamma$  at  $P$  give the gaps 10, 11, 12, 13 and 17 respectively.  $\square$

**Theorem 3.3.** *If  $i_2 = 7$ , then we have  $G(P) = \{1 \rightarrow 12, 15, \gamma_1, \gamma_2\}$  with  $13 \leq \gamma_1 < \gamma_2 \leq 29$  and  $\gamma_i \neq 15$  for  $i = 1, 2$  where if  $\gamma_1 = 13$  (resp. 14, resp. 16), then  $\gamma_2 \neq 28$  (resp. 26, 29, resp. 26, 27, 28), and if  $\gamma_1 \geq 17$ , then  $\gamma_2 \leq 25$ .*

*Proof.* Let  $\Gamma$  be a conic such that  $I(\Gamma \cap C, P) = 7$ . Then the intersection multiplicities of the curve  $C$  and the curves  $\Gamma + 2L_1$ ,  $\Gamma + L_1 + T$ ,  $\Gamma + 2T$ ,  $2\Gamma$  at  $P$  give the gaps 10, 11, 12 and 15 respectively. There are no cubics  $\Gamma'$  such that  $I(\Gamma' \cap C, P) \geq 10$ , because the intersection multiplicities of  $C$  and the cubic curves  $3L_0$ ,  $2L_0 + L_1$ ,  $L_0 + 2L_1$ ,  $3L_1$ ,  $2L_1 + T$ ,  $L_1 + 2T$ ,  $3T$ ,  $L_0 + \Gamma$ ,  $L_1 + \Gamma$ ,  $T + \Gamma$  at  $P$  are  $0 \rightarrow 9$  respectively. Thus, the intersection multiplicities of  $C$  and some two quartic curves at  $P$  give the two remaining gaps  $\gamma_1$  and  $\gamma_2$ .  $\square$

**Theorem 3.4.** *Let  $i_2 = 6$ .*

- i) *There exists a cubic  $\Gamma'$  such that  $I(C \cap \Gamma', P) = i_3$  with  $9 \leq i_3 \leq 20$ .*
- ii) *If  $13 \leq i_3 \leq 20$ , then  $G(P) = \{1 \rightarrow 11, 13, i_3 + 1, i_3 + 2, i_3 + 3\}$ .*
- iii) *If  $i_3 = 12$ , then  $G(P) = \{1 \rightarrow 11, 13, 14, 15, \gamma\}$  for some  $\gamma \in \{12, 16 \rightarrow 23, 25, 26, 27\}$ .*
- iv) *If  $i_3 = 11$ , then  $G(P) = \{1 \rightarrow 14, \gamma\}$  with  $15 \leq \gamma \leq 29$ .*
- v) *If  $i_3 = 9$  or  $10$ , then  $G(P) = \{1 \rightarrow 13, \gamma_1, \gamma_2\}$  where one of the following holds:*
  - (1)  $16 \leq \gamma_1 < \gamma_2 \leq 27$ , (2)  $\gamma_1 = 15$  and  $16 \leq \gamma_2 \leq 29$  with  $\gamma_2 \neq 28$ , (3)  $\gamma_1 = 14$  and  $15 \leq \gamma_2 \leq 29$ .

*Proof.* i) Let  $\Gamma$  be a conic such that  $I(\Gamma \cap C, P) = 6$ . Then the intersection multiplicities of the curve  $C$  and the curves  $\Gamma + L_1 + T$ ,  $\Gamma + 2T$ ,  $2\Gamma$  at  $P$  give the gaps 10, 11 and 13 at  $P$  respectively. We know that  $3L_0$ ,  $2L_0 + L_1$ ,  $L_0 + 2L_1$ ,  $3L_1$ ,  $2L_1 + T$ ,  $L_1 + 2T$ ,  $3T$ ,  $L_1 + \Gamma$ ,  $T + \Gamma$  are cubics whose

intersection multiplicities at  $P$  in  $C$  are  $0 \rightarrow 8$  respectively. Hence, there exists a cubic  $\Gamma'$  such that  $I(C \cap \Gamma', P) = i_3$  with  $9 \leq i_3 \leq 21$ . If  $i_3 = 21$ , then we have  $G(P) = \{0 \rightarrow 11, 13, 22, 23, 24\}$ . This is a contradiction.

- ii) If  $13 \leq i_3 \leq 20$ , then the intersection multiplicities of the curve  $C$  and the curves  $\Gamma' + L_0$ ,  $\Gamma' + L_1$  and  $\Gamma' + T$  at  $P$  give the gaps  $i_3 + 1$ ,  $i_3 + 2$  and  $i_3 + 3$  respectively, which are larger than 13.
- iii) The intersection multiplicities of the curve  $C$  and the curves  $\Gamma' + L_1$  and  $\Gamma' + T$  at  $P$  give the gaps 14 and 15 respectively. Using some quartic we get a gap  $\gamma \in \{12, 16 \rightarrow 23, 25, 26, 27\}$ .
- iv) The intersection multiplicities of the curve  $C$  and the curves  $\Gamma' + L_0$ ,  $\Gamma' + T$  at  $P$  give the gaps 12 and 14 respectively. Using some quartic we get a gap  $\gamma$  with  $15 \leq \gamma \leq 29$ .
- v) If  $i_3 = 9$  or  $10$ , then  $G(P)$  contains the set  $\{1 \rightarrow 13\}$ . Hence, the remaining two gaps  $\gamma_1$  and  $\gamma_2$  are induced by the intersection multiplicities of quartics and the curve  $C$ . Thus, we must have  $14 \leq \gamma_1 < \gamma_2 \leq 29$ . Moreover, the complement of the set  $\{1 \rightarrow 13, \gamma_1, \gamma_2\}$  in  $\mathbb{N}_0$  forms a subsemigroup, which imposes one of the conditions (1), (2), (3).  $\square$

**Theorem 3.5.** *Let  $i_2 = 5$ .*

- i) *There are two cubics  $\Gamma'$  and  $\Gamma''$  such that  $I(C \cap \Gamma', P) = i_{31}$  and  $I(C \cap \Gamma'', P) = i_{32}$  with  $8 \leq i_{31} < i_{32} \leq 21$ .*
- ii) *If  $i_{31} \geq 11$ , then we have  $i_{32} = i_{31} + 1$  and  $G(P) = \{1 \rightarrow 11, i_{31} + 1, i_{31} + 2, i_{31} + 3, i_{31} + 4\}$ . In this case,  $i_{31} \leq 19$ .*
- iii) *If  $i_{31} = 10$ , then  $G(P) = \{1 \rightarrow 14, \gamma\}$  for some  $\gamma$  with  $15 \leq \gamma \leq 29$ .*
- iv) *Let  $i_{31} = 9$ .*
  - (a) *In the case  $i_{32} \geq 12$ , we have  $G(P) = \{1 \rightarrow 12, i_{32} + 1, i_{32} + 2, i_{32} + 3\}$ .*
  - (b) *In the case  $i_{32} = 11$ ,  $G(P)$  is the same as in iii).*
  - (c) *In the case  $i_{32} = 10$ ,  $G(P)$  is the same as in Theorem 3.4 v).*
- v) *Let  $i_{31} = 8$ .*
  - (a) *In the case  $i_{32} \geq 11$ , then  $G(P) = \{1 \rightarrow 11, i_{32} + 1, i_{32} + 2, i_{32} + 3, \gamma\}$  where one of the following holds:*
    - (1) *If  $i_{32} = 11$ , then  $G(P)$  is the same as in iii),*
    - (2) *If  $i_{32} = 12$ , then  $G(P) = \{1 \rightarrow 11, 13, 14, 15, \gamma\}$  for some  $\gamma \in \{12, 16 \rightarrow 23, 25, 26, 27\}$ ,*
    - (3) *If  $i_{32} = 13$ , then  $G(P) = \{1 \rightarrow 11, 14, 15, 16, \gamma\}$  for some  $\gamma \in \{12, 13, 17 \rightarrow 23, 27, 28\}$ ,*
    - (4) *If  $i_{32} = 14$ , then  $G(P) = \{1 \rightarrow 11, 15, 16, 17, \gamma\}$  for some  $\gamma \in \{12, 13, 14, 18 \rightarrow 23, 29\}$ ,*
    - (5) *If  $i_{32} \geq 15$ , then  $G(P) = \{1 \rightarrow 11, i_{32} + 1, i_{32} + 2, i_{32} + 3, \gamma\}$  with  $12 \leq \gamma \leq 23$  and  $\gamma \neq i_{32} + 1, i_{32} + 2, i_{32} + 3$ .*
  - (b) *In the case  $i_{32} = 10$ , we have  $G(P)$  is the same as in Theorem 3.4 v).*
  - (c) *In the case  $i_{32} = 9$ , we have  $G(P) = \{1 \rightarrow 12, \gamma_1, \gamma_2, \gamma_3\}$  with  $13 \leq \gamma_1 < \gamma_2 < \gamma_3 \leq 29$  where one of the following holds:*
    - 1)  $\gamma_3 \leq 25$ , 2)  $(\gamma_1, \gamma_3) = (13, 26)$ , 3)  $(\gamma_1, \gamma_3) = (14, 27)$  or  $(13, 27)$ , 4)  $(\gamma_1, \gamma_2, \gamma_3) = (14, 15, 28)$  or  $(13, 14, 28)$ , 5)  $(\gamma_1, \gamma_2, \gamma_3) = (14, 16, 29)$  or  $(13, 15, 29)$  or  $(13, 14, 29)$  or  $(15, 16, 29)$ .

*Proof.* i) Let  $\Gamma$  be a conic such that  $I(\Gamma \cap C, P) = 5$ . Then the intersection multiplicities of the curve  $C$  and the curves  $\Gamma + 2T$ ,  $2\Gamma$  at  $P$  give the gaps 10 and 11 respectively. Hence the set  $G(P)$  of gaps contains  $\{1 \rightarrow 11\}$ . We know that  $3L_0$ ,  $2L_0 + L_1$ ,  $L_0 + 2L_1$ ,  $3L_1$ ,  $2L_1 + T$ ,  $L_1 + 2T$ ,  $3T$ ,  $T + \Gamma$  are cubics whose intersection multiplicities at  $P$  in  $C$  are  $0 \rightarrow 7$  respectively. Hence we have two cubics  $\Gamma'$  and  $\Gamma''$  as in i).

- ii) We have  $G(P) \supset \{i_{31} + 1, i_{31} + 2, i_{31} + 3\} \cup \{i_{32} + 1, i_{32} + 2, i_{32} + 3\}$ . So we must have  $i_{31} + 3 = i_{32} + 2$ , which implies that  $i_{32} = i_{31} + 1$ . If  $i_{31} = 20$ , then we get  $G(P) = \{1 \rightarrow 11, 21, 22, 23, 24\}$ . This contradicts the fact that  $H(P)$  is a semigroup.
- iii) We have  $G(P) \supseteq \{1 \rightarrow 13, i_{32} + 1, i_{32} + 2, i_{32} + 3\}$ . Hence, we must have  $i_{32} = 11$  or  $12$ . If  $i_{32} = 12$ ,

then  $G(P) = \{1 \longrightarrow 15\}$ . If  $i_{32} = 11$ , then  $G(P) = \{1 \longrightarrow 14, \gamma\}$  with some  $\gamma$  satisfying  $15 \leq \gamma \leq 29$ .

iv) We have  $G(P) \supseteq \{1 \longrightarrow 12, i_{32} + 1, i_{32} + 2, i_{32} + 3\}$ . If  $i_{32} \geq 12$ , the equality holds. If  $i_{32} = 11$ , then  $G(P)$  is the same as in iii). If  $i_{32} = 10$ , then  $G(P) = \{1 \longrightarrow 13, \gamma_1, \gamma_2\}$  where  $\gamma_1$  and  $\gamma_2$  satisfy one of the conditions (1), (2) and (3) in Theorem 3.4 v).

v) We have  $G(P) \supseteq \{1 \longrightarrow 11, i_{32} + 1, i_{32} + 2, i_{32} + 3\}$ . If  $i_{32} \geq 11$ , then  $G(P) = \{1 \longrightarrow 11, i_{32} + 1, i_{32} + 2, i_{32} + 3, \gamma\}$ . Since  $H(P)$  is an additive semigroup,  $\gamma$  satisfies one of the conditions in v)(a). If  $i_{32} = 10$ , then  $G(P)$  is the same as in Theorem 3.4 v). In the case  $i_{32} = 9$ , we have  $G(P) = \{1 \longrightarrow 12, \gamma_1, \gamma_2, \gamma_3\}$  with  $12 < \gamma_1 < \gamma_2 < \gamma_3$ . The complement  $H(P) = \mathbb{N}_0 \setminus G(P)$  becomes a subsemigroup if and only if  $\gamma_1, \gamma_2$  and  $\gamma_3$  satisfies one of the conditions in v) (c).  $\square$

### References

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