[研究論文] # Two dimensional affine toric varieties and Weierstrass semigroups

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Abstract

Let X be a 2-dimensional affine toric variety over an algebraically closed field k of characteritic 0. We consider a way to construct from X the Weierstrass semigroup H of a point on a complete non-singular curve over k such that the minimal embedding of its monomial curve Spec $k[H]$ into the affine space is derived from that of X by substituting monomials. For any $n \geqq 2$ we give examples of such X and H where H is generated by n elements.

Keywords: Affine toric variety, Weierstrass semigroup of a point

1. Introduction

A *numerical semigroup* H is a subsemigroup of the additive semigroup $\mathbb{N}_0$ of non-negative integers such that its complement in $\mathbb{N}_0$ is finite. Let $M(H)$ be the minimal set of generators for H. Then we can embed the monomial curve $X_H :=$ Spec $k[H]$ associated to H into the affine space through ${}^a\varphi_H : X_H \hookrightarrow \mathbb{A}^n$ defined by the k-algebra homomorphism $\varphi_H : k[X_1 \ldots, X_n] \longrightarrow k[H]$ sending X_i to t^{a_i} where $M(H) = \{a_1, \ldots, a_n\}$. We pose the following problem:

Let X be a 2-dimensional affine toric variety. Then find a numerical semigroup H with $n = \sharp M(H) = m-1$ such that the embedding ${}^a\varphi_H : X_H \hookrightarrow \mathbb{A}^n$ is derived from the minimal embedding $\iota : X \hookrightarrow \mathbb{A}^m$ by substituting monomials.

Why do we consider the above problem ? One of the reasons is that the above numerical semigroup H becomes *Weierstrass*[1)], i.e., there exist a complete non-singular irreducible algebraic curve C over k and its point P such that

$$H = \{n \in \mathbb{N}_0 | \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = nP\}$$

In Section 2 we describe a minimal set of generators for the semigroup S which defines the coordinate ring of a 2-dimensional affine toric variety. In Section 3 using the result of Section 2 we determine the minimal set of generators for the semigroup S concretely in several cases. In Section 4 we find numerical semigroups which are constructed from the 2-dimensional affine toric varieties corresponding to the semigroups S treated in Section 3.

2. The minimal set of generators

We set $T = \mathbb{G}_m^l$ where $\mathbb{G}_m$ is the multiplicative group Spec $k[X, X^{-1}]$. Let $M = \mathrm{Hom}_{Alg.Groups}(T, \mathbb{G}_m)$ and $N = \mathrm{Hom}_{Alg.Groups}(\mathbb{G}_m, T)$. We have a non-singular canonical pairing $< \ , \ >: M \times N \longrightarrow \mathbb{Z}$ where $\mathbb{Z}$ is the ring of intergers. We set $N_\mathbb{R} = N \bigotimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \bigotimes_\mathbb{Z} \mathbb{R}$. Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$, i.e., there are finite number of vectors $x_i \in N_\mathbb{R}$ defined over the ring $\mathbb{Q}$ of rational numbers such that

$$\sigma = \{\sum_{i=1}^{N'} \lambda_i x_i | \lambda_i \geq 0, \text{ all } i\}$$

and it contains no line through the origin. Then an *l-dimensional affine toric variety* is expressed by Spec $k[\check{\sigma} \cap M]$ where

$$\check{\sigma} = \{r \in M_\mathbb{R} | < r, a > \geq 0, \text{ all } a \in \sigma\}.$$

We devote this paper to 2-dimensional case.

From now on let Spec $k[\check{\sigma} \cap M]$ be a 2-dimensional affine toric variety where the interior of σ is non-empty. For such a σ we can find a basis e_1, e_2 of the $\mathbb{Z}$-module N such that

$$\sigma = \mathbb{R}_+ e_1 + \mathbb{R}_+(ae_1 + be_2) = \mathbb{R}_+(1,0) + \mathbb{R}_+(a,b),$$

where a and b are integers with $b > 0$ and $(a,b) = 1$[2]. If $b = 1$, then we may assume that $a = 0$. If $b > 1$, then we may assume that $0 < a < b$. The above cone σ is denoted by $\sigma_{a,b}$. In this section we prove that the semigroup $\check{\sigma}_{a,b} \cap M$ has a unique minimal set $M(a,b)$ of generators and it is described.

We can identify M with $\mathbb{Z}^2$ through the isomorphism sending a morphism $r : T \longrightarrow \mathbb{G}_m$ with $r(t_1,t_2) = t_1^{r_1} t_2^{r_2}$ to the element (r_1, r_2) of $\mathbb{Z}^2$. Let $S_{a,b}$ be the subsemigroup of $\mathbb{N}_0^2$ consisting of the elements (ν_1, ν_2) such that $-a\nu_1 + \nu_2$ is divisible by b. Then we have an isomorphism $S_{a,b} \longrightarrow \check{\sigma}_{a,b} \cap M$ of semigroups sending (ν_1,ν_2) to $(\nu_1,\nu_2)\begin{pmatrix} 1 & -a/b \\ 0 & 1/b \end{pmatrix}$.

Lemma 2.1. *Let $a > 0$. Then the semigroup $S_{a,b}$ is generated by*

$$(1,a), (0,b), (b,0) \text{ and } \left(\left[\frac{rb}{a}\right] + 1, a\left(\left[\frac{rb}{a}\right] + 1\right) - rb\right), 1 \leqq r < a,$$

where $[\ ]$ means the Gauss symbol.

Proof. We set

$$S_{a,b}^{+} = \cup_{n \in \mathbb{N}_0}\{(i, nb + ia) | i \geqq 0\} \text{ and } S_{a,b}^{-} = \cup_{n \in \mathbb{Z}_{<0}}\{(i, nb+ia) | i \geqq -\left[\frac{nb}{a}\right]\}.$$

Then we have $S_{a,b} = S_{a,b}^{+} \cup S_{a,b}^{-}$. If $(i, nb+ia) \in S_{a,b}^{+}$, then $(i, nb+ia) = i(1,a) + n(0,b)$, which implies that the semigroup $S_{a,b}^{+}$ is generated by $(1,a)$ and $(0,b)$. If $(i, nb+ia) \in S_{a,b}^{-}$, then

$$(i, nb+ia) = \left(-\left[\frac{nb}{a}\right], nb - a\left[\frac{nb}{a}\right]\right) + \left(i + \left[\frac{nb}{a}\right]\right)(1,a).$$

Let $n \in \mathbb{Z}_{<0}$. We set $-n = aq + r$ with $q \in \mathbb{N}_0$ and $0 \leqq r < a$. Then $\dfrac{b}{a}n = -bq - \dfrac{br}{a}$. If $r = 0$, then

$$\left(-\left[\frac{nb}{a}\right], nb - a\left[\frac{nb}{a}\right]\right) = q(b,0).$$

If $0 < r < a$, then $\dfrac{br}{a} \notin \mathbb{Z}$. Hence we get

$$\left[\frac{nb}{a}\right] = -bq - 1 - \left[\frac{br}{a}\right].$$

So, we obtain

$$\left(-\left[\frac{nb}{a}\right], nb - a\left[\frac{nb}{a}\right]\right) = q(b,0) + \left(-\left[\frac{-rb}{a}\right], -rb - a\left[\frac{-rb}{a}\right]\right).$$

Thus, we get our desired result. □

Theorem 2.2. *Let $a > 0$. Then the union G of the two sets $\{(1,a),(0,b),(b,0)\}$ and*

$$\left\{\left(\left[\frac{lb}{a}\right] + 1, a\left(\left[\frac{lb}{a}\right]+1\right) - lb\right) \middle| 1 \leqq l < a \text{ with } \nexists(m_1,\ldots,m_{l-1}) \in \mathbb{N}_0^{l-1}\right.$$

$$\left.\text{such that } l = \sum_{i=1}^{l-1} m_i i \text{ and } \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i\left(\left[\frac{ib}{a}\right]+1\right)\right\}$$

is a unique minimal set of generators for the semigroup $S_{a,b}$.

Proof. Let l be an integer with $1 \leqq l \leqq a-1$ and $(m_1,\ldots,m_{l-1}) \in \mathbb{N}_0^{l-1}$ such that

$$l = \sum_{i=1}^{l-1} m_i i \text{ and } \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i\left(\left[\frac{ib}{a}\right]+1\right).$$

Then

$$\left(\left[\frac{lb}{a}\right]+1, a\left(\left[\frac{lb}{a}\right]+1\right)-lb\right)=\sum_{i=1}^{l-1} m_i\left(\left[\frac{ib}{a}\right]+1, a\left(\left[\frac{ib}{a}\right]+1\right)-ib\right),$$

which belongs to

$$\left\langle\left(\left[\frac{ib}{a}\right]+1, a\left(\left[\frac{ib}{a}\right]+1\right)-ib\right)\right\rangle_{1\leqq i\leqq l-1}.$$

Hence, by Lemma 2.1 the semigroup $S_{a,b}$ is generated by the elements of the set G.

Since $0<a<b$ and $(a,b)=1$, any set of the sets $G\backslash\{(1,a)\}$, $G\backslash\{(0,b)\}$ and $G\backslash\{(b,0)\}$ is not a set of generators for $S_{a,b}$. We want to show that for any $l\in I_{a,b}$ the set $G\Big\backslash\left\{\left(\left[\frac{lb}{a}\right]+1, a\left(\left[\frac{lb}{a}\right]+1\right)-lb\right)\right\}$ is not a set of generators where we set

$$I_{a,b}=\{l\in\mathbb{Z}\mid 1\leqq l<a \text{ with } \nexists(m_1,\dots,m_{l-1})\in\mathbb{N}_0^{l-1}$$
$$\text{such that } l=\sum_{i=1}^{l-1} m_i i \text{ and } \left[\frac{lb}{a}\right]+1=\sum_{i=1}^{l-1} m_i\left(\left[\frac{ib}{a}\right]+1\right)\Bigg\}.$$

Take $l_0\in I_{a,b}$. Let

$$\left(\left[\frac{l_0b}{a}\right]+1, a\left(\left[\frac{l_0b}{a}\right]+1\right)-l_0b\right)$$
$$=m(b,0)+\sum_{l\in I_{a,b}\backslash\{l_0\}} m_l\left(\left[\frac{lb}{a}\right]+1, a\left(\left[\frac{lb}{a}\right]+1\right)-lb\right)+m'(1,a)+m''(0,b)$$

with m, m_l's, m' and $m''\in\mathbb{N}_0$. Then we must have

$$m=m'=m''=0 \text{ and } m_l=0 \text{ for } l>l_0.$$

Thus, we get $l_0\notin I_{a,b}$, which is a contradiction. Hence, G is a minimal set of generators for $S_{a,b}$.

Next, we shall show that a minimal set of generators for $S_{a,b}$ is unique. Let G' be another minimal set of generators. G' must contain $(0,b)$, because $(0,\nu_2)\in S_{a,b}$ implies that $b|\nu_2$. Moreover, $(1,a)$ should be in G', because $\nu_2\geqq a$ if $(1,\nu_2)\in S_{a,b}$. Since $(\nu_1,0)\in S_{a,b}$ implies that $b|\nu_1$, we have $(b,0)\in G'$. Let $(\nu_1,\nu_2)\in S_{a,b}$ with $1\leqq\nu_2\leqq a-1$. Then there exists l with $1\leqq l\leqq a-1$ such that $\nu_2=a\left(\left[\frac{lb}{a}\right]+1\right)-lb$. Then we get $\nu_1=\left[\frac{lb}{a}\right]+1+mb$ for some $m\in\mathbb{N}_0$. By induction on ν_2 we must have $G=G'$. □

Using the transformation from $S_{a,b}$ to $\check{\sigma}_{a,b}\cap\mathbb{Z}^2$ we obtain the following:

Corollary 2.3. i) *If $a=1$, then the set $\{(1,0),(0,1),(b,-1)\}$ is a unique minimal set of generators for the semigroup $\check{\sigma}_{1,b}\cap\mathbb{Z}^2$.*

ii) *Let $a\geqq 2$. Then the union of the two sets $\left\{(1,0),(0,1),(b,-a),\left(\left[\frac{b}{a}\right]+1,-1\right)\right\}$ and*

$$\left\{\left(\left[\frac{lb}{a}\right]+1,-l\right)\Bigg|\, 2\leqq l<a \textit{ with } \nexists(m_1,\dots,m_{l-1})\in\mathbb{N}_0^{l-1}\right.$$
$$\left.\textit{such that } l=\sum_{i=1}^{l-1} m_i i \textit{ and } \left[\frac{lb}{a}\right]+1=\sum_{i=1}^{l-1} m_i\left(\left[\frac{ib}{a}\right]+1\right)\right\}$$

is a unique minimal set of generators for the semigroup $\check{\sigma}_{a,b}\cap\mathbb{Z}^2$.

3. Examples of semigroups $\check{\sigma}_{a,b}\cap\mathbb{Z}^2$

In this section we determine the semigroups $\check{\sigma}_{a,b}\cap\mathbb{Z}^2$ whose minimal sets of generators consist of 4 elements. Moreover, using Corollary 2.3 we give examples of semigroups $\check{\sigma}_{a,b}\cap\mathbb{Z}^2$ whose minimal sets of generators consist of more than 4 elements.

Theorem 3.1. *Let $a\geqq 2$. Then the minimal set of generators for the semigroup $\check{\sigma}_{a,b}\cap\mathbb{Z}^2$ consists of 4 elements if and only if $b\equiv a-1 \bmod a$.*

Proof. By Corollary 2.3 it suffices to show that for any $2 \leqq l < a$ there exists

$$(m_1, \ldots, m_{l-1}) \in \mathbb{N}_0^{l-1} \text{ such that } l = \sum_{i=1}^{l-1} m_i i \text{ and } \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i \left(\left[\frac{ib}{a}\right] + 1\right)$$

if and only if $b \equiv a - 1 \bmod a$.

Let $b \equiv a - 1 \bmod a$. For any $2 \leqq i < a$ we have $\left[\frac{ib}{a}\right] = i\left[\frac{b}{a}\right] + i - 1$. If we set $m_1 = m_{l-1} = 1$ and $m_i = 0$ for i distinct from 1 and $l - 1$, then we have

$$\sum_{i=1}^{l-1} m_i i = 1 + l - 1 = l \text{ and } \sum_{i=1}^{l-1} m_i \left(\left[\frac{ib}{a}\right] + 1\right) = \left[\frac{b}{a}\right] + 1 + \left[\frac{(l-1)b}{a}\right] + 1 = l\left[\frac{b}{a}\right] + l = \left[\frac{lb}{a}\right] + 1.$$

Let $b \equiv a - j \bmod a$ for some $2 \leqq j \leqq a - 1$. We set $i_0 = \max\{i \in \mathbb{N}_0 \mid ij < a\} < \frac{a}{2}$. Let $l = i_0 + 1$. We assume that there exists

$$(m_1, \ldots, m_{l-1}) \in \mathbb{N}_0^{l-1} \text{ such that } l = \sum_{i=1}^{l-1} m_i i \text{ and } \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i \left(\left[\frac{ib}{a}\right] + 1\right).$$

Since $l = i_0 + 1$, by the definition of i_0 we have

$$l\left[\frac{b}{a}\right] + l - 1 = \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i \left(\left[\frac{ib}{a}\right] + 1\right) = \sum_{i=1}^{l-1} m_i \left(i\left[\frac{b}{a}\right] + i\right) = \left(\left[\frac{b}{a}\right] + 1\right) \sum_{i=1}^{l-1} m_i i = l\left(\left[\frac{b}{a}\right] + 1\right),$$

which is a contradiction. Thus, in this case the minimal set of generators for the semigroup $\check{\sigma}_{a,b} \cap \mathbb{Z}^2$ consists of at least 5 elements. □

Here, for any $n \geqq 4$ we give examples of semigroups $\check{\sigma}_{a,b} \cap \mathbb{Z}^2$ whose minimal set of generators consist of n elements.

Proposition 3.2. *If $a \geqq 2$ and $b \equiv 1$ mod a, then the minimal set of generators for the semigroup $\check{\sigma}_{a,b} \cap \mathbb{Z}^2$ consist of $a + 2$ elements as follows:*

$$\left\{(1,0), (0,1), (b,-a), \left(\left[\frac{b}{a}\right] + 1, -1\right), \left(2\left[\frac{b}{a}\right] + 1, -2\right), \ldots, \left((a-1)\left[\frac{b}{a}\right] + 1, -(a-1)\right)\right\}.$$

Proof. Since $b \equiv 1 \bmod a$, for any $2 \leqq i < a$ we obtain $\left[\frac{ib}{a}\right] = i\left[\frac{b}{a}\right]$. Let $2 \leqq l < a - 1$. Assume that there exists $(m_1, \ldots, m_{l-1}) \in \mathbb{N}_0^{l-1}$ such that

$$l = \sum_{i=1}^{l-1} m_i i \text{ and } \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i \left(\left[\frac{ib}{a}\right] + 1\right)$$

Then we have

$$l\left[\frac{b}{a}\right] + 1 = \left[\frac{lb}{a}\right] + 1 = \sum_{i=1}^{l-1} m_i \left(\left[\frac{ib}{a}\right] + 1\right) = \sum_{i=1}^{l-1} m_i \left(i\left[\frac{b}{a}\right] + 1\right) = \left[\frac{b}{a}\right] \sum_{i=1}^{l-1} m_i i + \sum_{i=1}^{l-1} m_i = l\left[\frac{b}{a}\right] + \sum_{i=1}^{l-1} m_i.$$

Since $\sum_{i=1}^{l-1} m_i \geqq 2$, this is a contradiction. By Corollary 2.3 we get our desired result. □

4. Numerical semigroups of 2-dimensional toric type

Let $\lambda \in \sigma_{a,b} \cap N$ such that $< r, \lambda > > 0$ for any non-zero $r \in \check{\sigma}_{a,b} \cap M$. We denote by $M(a,b)$ the minimal set of generators for the semigroup $\check{\sigma}_{a,b} \cap M$. Let $\sharp M(a,b) = n + 1$ and $M(a,b) = \{g_1, \ldots. g_{n+1}\}$. Let H be a numerical semigroup with minimal set $M(H) = \{a_1 \ldots, a_n\}$ of generators containing the semigroup $< \check{\sigma}_{a,b} \cap M, \lambda >$. Assume that for any $g \in M(a,b)$ we have a unique expression $< g, \lambda >= \nu_1 a_1 + \cdots + \nu_n a_n$ with $\nu_i \in \mathbb{N}_0$, all

i. Let $\mathbb{A}^l = \operatorname{Spec} k[X_1, \ldots, X_l]$ be the affine l-space. We can define the morpshim ${}^a\eta : \mathbb{A}^n \longrightarrow \mathbb{A}^{n+1}$ by the k-algebera homomorphism $\eta : k[Y_1, \ldots, Y_{n+1}] \longrightarrow k[X_1, \ldots, X_n]$ which sends Y_i to $\mathcal{X}^{<g_i,\lambda>} = X_1^{\nu_1} \cdots X_n^{\nu_n}$ where $< g_i, \lambda >= \nu_1 a_1 + \cdots + \nu_n a_n$ with $\nu_i \in \mathbb{N}_0$, all i. The above morphism ${}^a\eta : \mathbb{A}^n \longrightarrow \mathbb{A}^{n+1}$ is said to be *induced by* λ. Let ${}^a\pi : \operatorname{Spec} k[\check{\sigma}_{a,b} \cap M] = \operatorname{Spec} k[T^\mu]_{\mu \in \check{\sigma}_{a,b} \cap M} \longrightarrow \mathbb{A}^{n+1}$ is the embedding associated to the k-algebra homomorpshim π sending Y_i to T^{g_i} Moreover, we denote by $\varphi_H : k[X_1, \ldots, X_n] \longrightarrow k[H] = k[t^h]_{h \in H}$ the k-algebra homomorphism sending X_i to t^{a_i}, all i. We note that $\eta(\operatorname{Ker} \pi) \subseteqq \operatorname{Ker} \varphi_H$. The numerical semigroup H is *constructed from* $X_{a,b} = \operatorname{Spec} k[\check{\sigma}_{a,b} \cap M]$ and λ if $\operatorname{Spec} k[H]$ is isomorphic to the fiber product $\mathbb{A}^n \times_{\mathbb{A}^{n+1}} X_{a,b}$, i.e., the ideal $\operatorname{Ker} \varphi_H$ is generated by the elements of the set $\eta(\operatorname{Ker} \pi)$. In this case H is called a *numerical semigroup of* $2-$*dimensional toric type*. It is known that such a semigroup H is Weierstrass[1)].

Proposition 4.1. *A numerical semigroup H generated by 2 elements is of 2-dimensional toric type.*

Proof. Let H be generated by b and $b + n$ with $b, n \in \mathbb{N}_0$ satisfying $(b, n) = 1$. Take $\lambda = (b + n, b) \in \sigma_{1,b} \cap N$. We note that $M(1, b) = \{r_1 = (1, 0), r_2 = (0, 1), r_3 = (b, -1)\}$. Then we get $H =< \check{\sigma}_{a,b} \cap M, \lambda >$. Moreover, the morphism ${}^a\eta : \mathbb{A}^2 \longrightarrow \mathbb{A}^3$ induced by λ is associated to the k-algebra homomorphism $\eta : k[Y_1, Y_2, Y_3] \longrightarrow k[X_1, X_2]$ sending Y_1, Y_2 and Y_3 to X_2, X_1 and X_1^{b+n-1} respectively. Since the kernel $\operatorname{Ker} \varphi_H$ of φ_H is generated by the element $\eta(Y_2 Y_3 - Y_1^b) = X_1^{b+n} - X_2^b$ with $Y_2 Y_3 - Y_1^b \in \operatorname{Ker} \pi$, the numerical semigroup H is of 2-dimensional toric type. □

We study numerical semigroups which will be constructed from 2-dimensional affine toric varieties $\operatorname{Spec} k[\check{\sigma}_{a,b} \cap M]$ where a and b are as in Theorem 3.1.

Lemma 4.2. *Let $a \geqq 2$ and $b > a$ with $b \equiv a - 1 \bmod a$. Take $n \geqq 1$ such that $(b, n) = 1$. Let H be the semigroup generated by $a_1 = b, a_2 = b + na$ and $a_3 = \left(n + \left[\frac{b}{a}\right]\right) b + n$. Then the following hold:*

i) *$\{a_1, a_2, a_3\}$ is the minimal set of generators for H.*

ii) *We have $\alpha_1 := \min\{\alpha > 0 | \alpha a_1 \in \langle a_2, a_3 \rangle\} = (n-1)a + b + 1$, $\alpha_2 := \min\{\alpha > 0 | \alpha a_2 \in \langle a_1, a_3 \rangle\} = \left[\frac{b}{a}\right] + 1$ and $\alpha_3 := \min\{\alpha > 0 | \alpha a_3 \in \langle a_1, a_2 \rangle\} = a$ where $\langle c, d \rangle$ is the semigroup generated by positive integers c and d. In fact, we have relations*

$$((n-1)a + b + 1)a_1 = \left[\frac{b}{a}\right] a_2 + (a-1)a_3, \ \left(\left[\frac{b}{a}\right] + 1\right) a_2 = a_1 + a_3, \ a \cdot a_3 = ((n-1)a + b)a_1 + a_2.$$

Proof. i) Since $b \equiv a - 1 \bmod a$ and $(b, n) = 1$, H is a numerical semigroup. We note that $a_1 < a_2 < a_3$. Let $a_3 = \mu a_1 + \nu a_2$ with $\mu, \nu \in \mathbb{N}_0$. In view of $(b, n) = 1$, we have $a\nu = bq + 1$ with $q \geqq 1$ and $\nu \geqq \left[\frac{b}{a}\right] + 1$. Hence, we get

$$nb + \left[\frac{b}{a}\right] b + n \geqq \mu b + \left(\left[\frac{b}{a}\right] + 1\right)(b + na) = \mu b + b + nb + n + \left[\frac{b}{a}\right] b$$

because of $b \equiv a - 1 \bmod a$. This is a contradiction.

ii) First, we prove that $\alpha_2 = \left[\frac{b}{a}\right] + 1$. Assume that $\alpha_2 < \left[\frac{b}{a}\right] + 1$. Let $\alpha_2 a_2 = \nu a_1 + \mu a_3 < a_1 + a_3$ If $\nu = 0$, then $\mu = 1$, which contradicts i). Hence, $\mu = 0$, which implies that $b | \alpha_2$. Thus, we get $b \leqq \alpha_2 < \left[\frac{b}{a}\right] + 1 \leqq \frac{b}{2} + 1$, which is a contradiction. Therefore, we have $\alpha_2 = \left[\frac{b}{a}\right] + 1$.

Second, we assume that $\alpha_1 a_1 = \alpha_3 a_3$. Then $b | \alpha_3$. Hence, $b \leqq \alpha_3 \leqq a$, which is a contradiction.

Third, we assume that $\alpha_2 a_2 = \alpha_3 a_3$. Then $\left(\left[\frac{b}{a}\right] + 1\right) a - \alpha_3$ is divisible by b. Hence, we get

$$b \leqq \left(\left[\frac{b}{a}\right] + 1\right) a - \alpha_3 = \left[\frac{b}{a}\right] a + a - 1 - \alpha_3 + 1 = b - \alpha_3 + 1 \leqq b - 1,$$

which is a contradiction. It is easy to check that $\alpha_1 a_1 \neq \alpha_2 a_2$.

We have relations

$$\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3, \ \left(\left[\frac{b}{a}\right] + 1\right) a_2 = a_1 + a_3, \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2.$$

By 3) we obtain $\alpha_1 = 1 + \alpha_{31}$, $\left[\frac{b}{a}\right] + 1 = \alpha_2 = \alpha_{12} + \alpha_{32}$ and $\alpha_3 = \alpha_{13} + 1$ with $\alpha_{31} > 0$, $\alpha_{12} > 0$, $\alpha_{32} > 0$ and $\alpha_{13} > 0$. Consider $\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3$. Then we have $\alpha_{12} a + \alpha_{13} = bq$ with $q \geqq 1$. Assume that $\alpha_{13} \geqq a$. Then we have

$$\alpha_1 a_1 \geqq a(\left(n + \left[\frac{b}{a}\right]\right) b + n) = ((n-1)a + b + 1)b + na > ((n-1)a + b + 1)b,$$

which is a contradiction. Hence, we may assume that $0 \leqq \alpha_{13} \leqq a - 1$, which implies that in the case $q \geqq 2$ we get $\alpha_{12} \geqq \left[\frac{b}{a}\right] + 1$. Since $\left[\frac{b}{a}\right] + 1 = \alpha_{12} + \alpha_{32}$, we must have $q = 1$. Therefore, we get $\alpha_{12} = \left[\frac{b}{a}\right]$ and $\alpha_{13} = a - 1$, which implies that $\alpha_1 = (n-1)a + b + 1$ and $\alpha_3 = a$. □

Using Lemma 4.2 and Theorem 3.1 we get numerical semigroups of 2-dimensional toric type which are generated by three elements.

Proposition 4.3. *Let $a \geqq 2$ and $b > a$ with $b \equiv a - 1 \bmod a$. Take $n \geqq 1$ such that $(b, n) = 1$. Then the semigroup H generated by $a_1 = b$, $a_2 = b + na$ and $a_3 = \left(n + \left[\frac{b}{a}\right]\right) b + n$ is of 2-dimensional toric type.*

Proof. Take $\lambda = (na + b, b) \in \sigma_{a,b} \cap N$. By Theorem 3.1 we have

$$M(a,b) = \left\{ r_1 = (1,0), r_2 = (0,1), r_3 = (b, -a), r_4 = \left(\left[\frac{b}{a}\right] + 1, -1\right) \right\}.$$

Then we get $H = < \check{\sigma}_{a,b} \cap M, \lambda >$. Moreover, the morphism ${}^a\eta : \mathbb{A}^3 \longrightarrow \mathbb{A}^4$ induced by λ is associated to the k-algebra homomorphism $\eta : k[Y_1, Y_2, Y_3, Y_4] \longrightarrow k[X_1, X_2, X_3]$ sending Y_1, Y_2, Y_3 and Y_4 to $X_2, X_1, X_1^{(n-1)a+b}$ and X_3 respectively. Since Ker φ_H is generated by the elements

$$\eta(Y_2 Y_3 - Y_1^{[\frac{b}{a}]} Y_4^{a-1}) = X_1^{(n-1)a+b+1} - X_2^{[\frac{b}{a}]} X_3^{a-1}, \eta(Y_1^{[\frac{b}{a}]+1} - Y_2 Y_4) = X_2^{[\frac{b}{a}]+1} - X_1 X_3$$

$$\text{and } \eta(Y_4^a - Y_1 Y_3) = X_3^a - X_1^{(n-1)a+b} X_2$$

with

$$Y_2 Y_3 - Y_1^{[\frac{b}{a}]} Y_4^{a-1}, Y_1^{[\frac{b}{a}]+1} - Y_2 Y_4, Y_4^a - Y_1 Y_3 \in \text{Ker } \pi,$$

the numerical semigroup H is of 2-dimensional toric type. □

We investigate numerical semigroups generated by a large number of elements. The results which we get will be applied to numerical semigroups constructed from 2-dimensional affine toric varieties.

Lemma 4.4. *Let b and c be positive integers such that $(b, c) = 1$. Let H be the semigroup generated by $a_0 = b$, $a_1 = b + c$, $a_2 = b + 2c, \ldots, a_{b-1} = b + (b-1)c$. Let $\varphi_H : k[X_0, X_1, \ldots, X_{b-1}] \longrightarrow k[H] = k[t^h]_{h \in H}$ be the k-algebra homomorphism sending X_i to t^{a_i}, all i. Then we have the following:*
i) *The minimal set $M(H)$ of generators for H is $\{a_0, a_1, a_2, \ldots, a_{b-1}\}$.*
ii) *The ideal $I_H =$ Ker φ_H is generated by*

$$X_0^{c+1} X_{i-1} - X_i X_{b-1} \ (1 \leqq i \leqq b-1) \text{ and } X_i X_j - X_{i-1} X_{j+1} \ (1 \leqq i \leqq j \leqq b-2).$$

Proof. i) Let $2 \leqq i \leqq b-1$. Assume that $b + ic = \nu b + \sum_{j=1}^{i-1} \nu_j (b + jc)$, ν, ν_j's $\in \mathbb{N}_0$. Then we get $ic \equiv \left(\sum_{j=1}^{i-1} \nu_j j\right) c \bmod b$. In view of $(b, c) = 1$, we have $i = \sum_{j=1}^{i-1} \nu_j j + \mu b$, $\mu \in \mathbb{Z}$. Hence, we get $\nu + \sum_{j=1}^{i-1} \nu_j = 1 + \mu c$ with $\mu \geqq 1$ because of $\sum_{j=1}^{i-1} \nu_j \geqq 2$. Thus, we get

$$b + ic \geqq \nu b + \sum_{j=1}^{i-1} \nu_j (b + c) = \nu b + (1 + \mu c - \nu)(b + c) = (1 + \mu c) b + (1 + \mu c - \nu) c,$$

which implies that $b-1 \geqq i \geqq \mu b + (1+\mu c - \nu) \geqq b+2$. This is a contradiction.

ii) Let J be the ideal generated by the above polynomials. It is trivial that $J \subseteq I_H$. It suffices to show that any polynomial $f = \prod_{i=0}^{b-1} X_i^{\nu_i} - \prod_{i=0}^{b-1} X_i^{\mu_i} \in I_H$ with $\nu_i\mu_i = 0$, all i, is contained in J. In view of $X_iX_j \equiv X_{i-1}X_{j+1} \bmod J$ $(1 \leqq i \leqq j \leqq b-2)$ we may assume that $\nu_0 > 0$ and $\mu_{b-1} > 0$, because otherwise we may decrease the degree of f. By $X_{b-1}^2 \equiv X_0^{c+1}X_{b-2} \bmod J$, we may assume that $\prod_{i=0}^{b-1} X_i^{\mu_i} = X_iX_{b-1}$ with $1 \leqq i \leqq b-2$. In view of $X_0^{c+1}X_{i-1} \equiv X_iX_{b-1} \bmod J$ for any i with $1 \leqq i \leqq b-1$ we may decrease the degree of f. □

Using the above lemma and Proposition 3.2 we obtain numerical semigroups of 2-dimensional toric type which are generated by r elements for any $r \geqq 4$.

Proposition 4.5. *Let $a \geqq 2$ and $n \geqq 1$ with $(n, a+1) = 1$. We denote by H the semigroup generated by $a_0 = a+1, a_1 = a+1+na, a_2 = a+1+2na, \dots, a_a = a+1+a\cdot na$. Then H is of 2-dimensional toric type with $\sharp M(H) = a+1$*

Proof. In view of $(n, a+1) = 1$ we have $M(H) = \{a_0, a_1, \dots, a_a\}$ by Lemma 4.4. We denote by φ_H the k-algebra homomorphism as in Lemma 4.4. Take $\lambda = (na+a+1, a+1) \in \sigma_{a,a+1} \cap N$. By Proposition 3.2 we have $M(a, a+1) = \{r_0 = (0,1)\} \cup \{r_{l+1} = (l+1, -l) | l = 0, \dots, a\}$. We note that $\langle r_i, \lambda \rangle = a_i$ for $0 \leqq i \leqq a$ and $\langle r_{a+1}, \lambda \rangle = a+1+(a+1)\cdot na$. Hence, we get $\langle \check{\sigma}_{a,a+1} \cap M, \lambda \rangle = H$. Moreover, the morphism

$$^{a}\eta : \mathbb{A}^{a+1} = \mathrm{Spec}\,[X_0, X_1, \dots, X_a] \longrightarrow \mathbb{A}^{a+2} = \mathrm{Spec}\,[Y_0, Y_1, \dots, Y_{a+1}]$$

induced by λ is associated to the k-algebra homomorphism η sending Y_i $(0 \leqq i \leqq a)$ and Y_{a+1} to X_i $(0 \leqq i \leqq a)$ and X_0^{na+1} respectively. By Lemma 4.4 Ker φ_H is generated by the elements

$$\eta(Y_{a+1}Y_{i-1} - Y_iY_a) = X_0^{na+1}X_{i-1} - X_iX_a \ (1 \leqq i \leqq a)$$

$$\text{and } \eta(Y_iY_j - Y_{i-1}Y_{j+1}) = X_iX_j - X_{i-1}X_{j+1} \ (1 \leqq i \leqq j \leqq a-1)$$

with $Y_{a+1}Y_{i-1} - Y_iY_a, Y_iY_j - Y_{i-1}Y_{j+1} \in \mathrm{Ker}\,\pi$. Hence, the numerical semigroup H is of 2-dimensional toric type. □

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