[研究論文] The numerical semigroup of toric type at a ramification point on a double covering of a curve

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Abstract

Let H be a numerical semigroup of toric type, i.e., a minimal embedding of its monomial curve into the affine space is derived from an embedding of some affine toric variety by substituting monomials. In this paper we mainly treat numerical semigroups H generated by three elements. In some cases we investigate whether there is a double covering $\pi : \tilde{C} \longrightarrow C$ of a curve over an algebraically closed field k of characteristic 0 with a ramification point $\tilde{P}$ whose Weiertrass semigroup $H(\tilde{P})$ is of toric type with $H(\pi(\tilde{P})) = H$.

Keywords: Numerical semigroup, Affine toric variety, Double covering of a curve

1. Introduction

A *numerical semigroup* H is a submonoid of the additive monoid $\mathbb{N}_0$ of non-negative integers such that its complement $\mathbb{N}_0 \backslash H$ in $\mathbb{N}_0$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. We define a map d_2 from the set of numerical semigroups into the same set by sending $\tilde{H}$ to the numerical semigroup

$$d_2(\tilde{H}) = \left\{ \frac{\tilde{h}}{2} \,\middle|\, \tilde{h} \in \tilde{H} \text{ is even} \right\}. \tag{1}$$

Let C be a curve where a *curve* means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0 in this paper. For a point P of C we define

$$H(P) = \{n \in \mathbb{N}_0 \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = nP\}, \tag{2}$$

which is called the *Weierstrass semigroup* of P. Then $H(P)$ is a numerical semigroup. We consider a double covering $\pi : \tilde{C} \longrightarrow C$ of a curve with a ramification point $\tilde{P}$. Then we have $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. A numerical semigroup $\tilde{H}$ is said to be of *double covering type* if there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = \tilde{H}$.

Let $M(H) = \{a_1, \ldots, a_n\}$ be the minimal set of generators for a numerical semigroup H. Then we can embed the monomial curve $C_H :=$ Spec $k[H]$ associated to H into the n-dimensional affine space through ${}^a\varphi_H : C_H \hookrightarrow \mathbb{A}^n$ defined by the k-algebra homomorphism $\varphi_H : k[X_1 \ldots, X_n] \longrightarrow k[H] = k[t^h]_{h \in H}$ sending X_i to t^{a_i} for each i. The numerical semigroup H is said to be *of toric type* if there exists an affine toric variety X of dimension $m+1-n$ embedded in the m-dimensional affine space $\mathbb{A}^m$ such that we have the fiber product

$$\begin{array}{ccc} C_H & \overset{{}^a\varphi_H}{\hookrightarrow} & \mathbb{A}^n \\ \downarrow & \square & \downarrow^{\zeta} \\ X & \hookrightarrow & \mathbb{A}^m \end{array}$$

of X and $\mathbb{A}^n$ over $\mathbb{A}^m$ through some morphism ζ from $\mathbb{A}^n$ to $\mathbb{A}^m$ sending $(x_1, \ldots, x_n)$ to

$$(M_1(x_1, \ldots, x_n), \ldots, M_m(x_1, \ldots, x_n))$$

where $M_i(x_1, \ldots, x_n)$'s are non-constant monomials.

In Section 2 we determine $d_2(\tilde{H})$ for a numerical semigroup $\tilde{H}$ generated by two elements. For a numerical semigroup $\tilde{H}$ generated by three elements its image by d_2 is given in Section 3. In Section 4 we give examples $\tilde{H}$ which are of double covering type in each case of Sections 2 and 3. But in one case there are $\tilde{H}$ which are not of double covering type.

2. On numerical semigroups $d_2(\tilde{H})$ with $\tilde{H}$ generated by two elements

In this section we will show that $d_2(\tilde{H})$ is generated by two or three elements if $\tilde{H}$ is generated by two elements. We use the following notation: for an m-semigroup H, i.e., the least positive integer in H is m, we set $S(H) = \{m, s_1, \ldots, s_{m-1}\}$ where $s_i = \min\{h \in H \mid h \equiv i \mod m\}$, which is called the *standard basis* for H.

Lemma 2.1. *Let a and b be positive integers with $a < b$ and $(a, b) = 1$. Assume that either a or b is even. We set $\tilde{H} = \langle a, b\rangle$. Then $d_2(\tilde{H})$ is generated by two elements. In fact, if a (resp. b) is even, then $d_2(\tilde{H}) = \langle \frac{a}{2}, b\rangle$ (resp. $d_2(\tilde{H}) = \langle a, \frac{b}{2}\rangle$).*

Proof. Assume that a is even. Then we have $d_2(\tilde{H}) \supseteqq \langle \frac{a}{2}, b\rangle$. Let $\tilde{h} \in \tilde{H}$, which implies that $\tilde{h} = na + mb$ with n, $m \in \mathbb{N}_0$. If $\tilde{h}$ is even, then m is even. Hence, $\frac{\tilde{h}}{2} = n \cdot \frac{a}{2} + \frac{m}{2} \cdot b$. Thus, we get $d_2(\tilde{H}) \subseteqq \langle \frac{a}{2}, b\rangle$. In the case where b is even the above way works well. □

Lemma 2.2. *Let a and b be positive integers with $a < b$ and $(a, b) = 1$. Assume that a and b are odd. We set $\tilde{H} = \langle a, b\rangle$. Then $d_2(\tilde{H})$ is generated by three elements. In fact, we have $d_2(\tilde{H}) = \langle a, \frac{a+b}{2}, b\rangle$.*

Proof. Since a and b are odd, we get $d_2(\tilde{H}) \supseteqq \langle a, \frac{a+b}{2}, b\rangle$. Let $\tilde{h} \in \tilde{H}$ be even. Then $\tilde{h} = na + mb$ with n, $m \in \mathbb{N}_0$ where both n and m are even or odd. If n and m are even, then $\frac{\tilde{h}}{2} = \frac{n}{2} \cdot a + \frac{m}{2} \cdot b$. If n and m are odd with $n \leqq m$ (resp $n > m$), then

$$\frac{\tilde{h}}{2} = n \cdot \frac{a+b}{2} + \frac{m-n}{2} \cdot b \text{ (resp. } \frac{\tilde{h}}{2} = \frac{n-m}{2} \cdot a + m \cdot \frac{a+b}{2}\text{)}. \tag{3}$$

□

We know that every numerical semigroup generated by two or three elements are of toric type[1]. Hence by Lemmas 2.1 and 2.2 we get the following:

Proposition 2.3. *If $\tilde{H}$ is a numerical semigroup generated by two elements, then $\tilde{H}$ and $d_2(\tilde{H})$ are of toric type.*

3. On numerical semigroups $d_2(\tilde{H})$ with $\tilde{H}$ generated by three elements

We will investigate the numerical semigroups $d_2(\tilde{H})$ with $\tilde{H}$ generated by three elements in this section.

Lemma 3.1. *Let $\tilde{H}$ be a numerical semigroup generated by three positive integers a, b and c. Assume that two of a, b and c are even. Then the numerical semigroup $d_2(\tilde{H})$ is generated by three elements. In fact, if a and b are even, then $d_2(\tilde{H}) = \langle \frac{a}{2}, \frac{b}{2}, c\rangle$.*

Proof. We may assume that a and b are even. Then $d_2(\tilde{H}) \supseteqq \langle \frac{a}{2}, \frac{b}{2}, c\rangle$. Let $\tilde{h} = na + mb + lc$ be an even number with n, m and $l \in \mathbb{N}_0$. Then l must be even. Hence, $\frac{\tilde{h}}{2} = n \cdot \frac{a}{2} + m \cdot \frac{b}{2} + \frac{l}{2} \cdot c$. □

Thus, we get the following:

Proposition 3.2. *Let $\tilde{H}$ be a numerical semigroup generated by three positive integers a, b and c. Assume that two of a, b and c are even. Then $\tilde{H}$ and $d_2(\tilde{H})$ are of toric type.*

Lemma 3.3. *Let $\tilde{H}$ be a numerical semigroup generated by three positive integers a, b and c. Assume that only one of a, b and c is even. Then the numerical semigroup $d_2(\tilde{H})$ is generated by four elements. In fact, if a is even, then $d_2(\tilde{H}) = \langle \frac{a}{2}, b, c, \frac{b+c}{2} \rangle$.*

Proof. We may assume that a is even. Then $d_2(\tilde{H}) \supseteqq \langle \frac{a}{2}, b, c, \frac{b+c}{2} \rangle$. Let $\tilde{h} = na+mb+lc$ be an even number with n, m and $l \in \mathbb{N}_0$. Then $mb+lc$ is even. Since b and c are odd, by the proof of Lemma 2.2 we obtain $\frac{mb+lc}{2} \in \langle b, c, \frac{b+c}{2} \rangle$. Thus, we get $d_2(\tilde{H}) \subseteqq \langle \frac{a}{2}, b, c, \frac{b+c}{2} \rangle$. □

Example 3.4. Let n be a positive integer. We set $\tilde{H} = \langle 8, 8n+1, 8n+3 \rangle$. Then $d_2(\tilde{H}) = \langle 4, 8n+1, 8n+3, 8n+2 \rangle$. In this case, $\tilde{H}$ and $d_2(\tilde{H})$ are of toric type.

Proof. It sufficient to show that $H = d_2(\tilde{H})$ is of toric type. We set $a_1 = 4$, $a_2 = 8n+1$, $a_3 = 8n+3$ and $a_4 = 8n+2$. For each i we denote by α_i

$$\min\{\alpha \in \mathbb{N}_0 > 0 \mid \alpha a_i \in \langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_4 \rangle\}.$$

Then we get $\alpha_1 = 4n+1$ and $\alpha_2 = \alpha_3 = \alpha_4 = 2$. In fact, we have the following relations:

$$\alpha_1 a_1 = a_2 + a_3, \tag{4}$$

$$\alpha_2 a_2 = 2n a_1 + a_4, \tag{5}$$

$$\alpha_3 a_3 = (2n+1) a_1 + a_4, \tag{6}$$

$$\alpha_4 a_4 = a_2 + a_3. \tag{7}$$

Among the coefficients of the above equalities we have a relation $\begin{vmatrix} 4n+1 & -1 & -1 \\ -2n & 2 & 0 \\ -2n-1 & 0 & 2 \end{vmatrix} = 8n+2 = a_4$. Hence H is of toric type[1)]. □

Lemma 3.5. *Let $\tilde{H}$ be a numerical semigroup generated by three positive integers a, b and c. Assume that a, b and c are odd. Then the numerical semigroup $d_2(\tilde{H})$ is generated by six elements. In fact, we have $d_2(\tilde{H}) = \langle a, b, c, \frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2} \rangle$.*

Proof. It is obvious that $d_2(\tilde{H}) \supseteqq \langle a, b, c, \frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2} \rangle$. Let $\tilde{h} = na+mb+lc$ be an even number with n, m and $l \in \mathbb{N}_0$. If n, m and l are even, then $\frac{\tilde{h}}{2} \in \langle a, b, c \rangle$. Otherwise, we may assume that n is even and that both m and l are odd numbers with $m \leqq l$. Then we get

$$\frac{\tilde{h}}{2} = \frac{n}{2} \cdot a + m \cdot \frac{b+c}{2} + \frac{l-m}{2} \cdot c. \tag{8}$$

□

Here, we give examples satisfying the assumptions in Lemma 3.5.

Example 3.6. Let a be an odd number with $a \geqq 7$. We set $\tilde{H} = \langle a, a+2, a+6 \rangle$. Then we have

$$d_2(\tilde{H}) = \langle a, a+1, a+2, a+3, a+4, a+6 \rangle. \tag{9}$$

4. Numerical semigroups of double covering type and toric type

In this section we are interested in numerical semigroups treated in Sections 2 and 3 which are of double covering type.

Lemma 4.1. *Let $\tilde{H}$ be a numerical semigroup generated by two positive integers a and b with $(a, b) = 1$. Then there is a cyclic covering of the projective line $\mathbb{P}^1$ of degree a with a totally ramification point $\tilde{P}$ such that $H(\tilde{P}) = \tilde{H}$.*

Proof. We consider two variables x and y over k satisfying an equation $y^a = \prod_{i=1}^{b}(x - c_i)$ where c_i's are distinct elements of k. Let $\mathbb{P}^1$ be the projective line with function field $k(x)$ and C a curve with function field $k(x, y)$. Then we have a cyclic covering $f : C \longrightarrow \mathbb{P}^1$ corresponding to the inclusion $k(x) \subset k(x, y)$, i.e., for a point P of C the map f send P to the projective coordinate $(1, x(P))$. Then by Riemann-Hurwitz formula we obtain that the genus of C is equal to $\dfrac{(a-1)(b-1)}{2}$. Let $P_\infty \in C$ with $f(P_\infty) = (0 : 1)$ and $P_i \in C$ with $f(P_i) = (1 : c_i)$ for all i. Then P_∞ is a total ramification point of f. Moreover, we have

$$(x) = (x)_0 - aP_\infty \tag{10}$$

and

$$(y) = \sum_{i=1}^{b} P_i - bP_\infty. \tag{11}$$

Here (x) and (y) are the divisors of the functions x and y respectively, and $(x)_0$ denotes the divisor of the zeros of x. Hence we have $H(P_\infty) \supseteqq \langle a, b\rangle$. In view of $g(H(P_\infty)) = \dfrac{(a-1)(b-1)}{2}$ we get $H(P_\infty) = \langle a, b\rangle$. □

By Lemmas 4.1 we get numerical semigroups $\tilde{H}$ as in Lemma 2.1 which are of double covering type as follows:

Proposition 4.2. *Let a and b be relatively prime positive integers such that b is odd. Then the numerical semigroup $\tilde{H} = \langle 2a, b\rangle$ is of double covering type and toric type. Moreover, $d_2(\tilde{H}) = \langle a, b\rangle$ is a numerical semigroup of toric type generated by two elements.*

We can construct the following numerical semigroup satisfying the condition in Lemma 2.2 which is of double covering type:

Example 4.3. Let $\tilde{H} = \langle 3, 5\rangle$. By Lemma 2.2 we have $d_2(\tilde{H}) = \langle 3, 4, 5\rangle$. We will show that $\tilde{H}$ is of double covering type. The following proof is due to Takeshi Harui, Akira Ohbuchi and myself. Let x and y be variables over the algebraically closed field k of characteristic 0 such that $y^3 = (x - c_1)(x - c_2)(x - c_3)^2$ where c_1, c_2 and $c_3 \in k$ are distinct. Let $\pi : C \longrightarrow \mathbb{P}^1$ be the three-sheeted covering of the projective line corresponding to the inclusion $k(x) \subset k(x, y)$, i.e., π sends P to the homogeneous coordinate $(1, x(P))$. Let P_1, P_2, P_3 and P_4 be the ramification points of π. Then C is of genus 2. We note that P_i's are non-Weierstrass points. Let ι be the hyperelliptic involution on C and σ the automorphism on C such that $C/\langle\sigma\rangle \cong \mathbb{P}^1$. Then $\sigma \circ \iota = \iota \circ \sigma$. Hence, we may assume that $\iota P_1 = P_2$, which implies that $P_1 + P_2 \sim g_2^1$. We set $D = 2P_1 - P_2$. Then we have $2D = 4P_1 - 2P_2 \sim P_1 + 3P_2 - 2P_2 = P_1 + P_2$. We set $\mathcal{L} = \mathcal{O}_C(-D)$. Then there is an isomorphism $\mathcal{L}^{\otimes 2} \overset{\phi}{\cong} \mathcal{O}_C(-(P_1 + P_2)) \subset \mathcal{O}_C$. Hence, the direct sum $\mathcal{O}_C \oplus \mathcal{L}$ has an $\mathcal{O}_C$-algebra structure through ϕ. Let $\pi : \tilde{C} = \mathrm{Spec}\ (\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ be a canonical morphism, which is a double covering whose branch points are P_1 and P_2. We call the above way to construct a double covering the *Mumford's method.* Let $\tilde{P}_i$ be the ramification point of π over P_i for $i = 1, 2$. Then by Proposition 2.1 in 3) we have

$$h^0(2\tilde{P}_1) = h^0(P_1) + h^0(-P_1 + P_2) = 1 \tag{12}$$

and

$$h^0(4\tilde{P}_1) = h^0(2P_1) + h^0(P_2) = 2. \tag{13}$$

In view of $H(P_1) \not\ni 2$ we get $3 \in H(\tilde{P}_1)$. Moreover, we obtain

$$h^0(6\tilde{P}_1) = h^0(3P_1) + h^0(P_1 + P_2) = 4, \tag{14}$$

because of $P_1 + P_2 \sim g_2^1$. Hence, $3, 5 \in H(\tilde{P}_1)$, which implies that $H(\tilde{P}_1) = \langle 3, 5\rangle$. Thus, $\tilde{H} = \langle 3, 5\rangle$ is of double covering type.

Here, we give some notations. For a numerical semigroup H we set $c(H) = \min\{h \in \mathbb{N}_0 \mid h + \mathbb{N}_0 \subseteqq H\}$. It is well-known that $c(H) \leqq 2g(H)$ (for example, see Lemma 2.1 (3) in 4)). A numerical semigroup H is said to be *symmetric* if

$c(H) = 2g(H)$. For some numerical semigroups $\tilde{H}$ with $\sharp M(\tilde{H}) = 3$ we have the following numerical semigroups $\tilde{H}$ of double covering type as in Lemma 3.1.

Proposition 4.4. *Let a and b be relatively prime positive integers with $3 \leqq a < b$. Let c be an odd integer with $c \geqq 2(a-1)(b-1)-1$. We set $\tilde{H} = \langle 2a, 2b, c\rangle$. Then $\tilde{H}$ is of double covering type with $d_2(\tilde{H}) = \langle a, b\rangle$. In this case, $\tilde{H}$ and $H = d_2(\tilde{H})$ are of toric type.*

Proof. By Lemma 3.1 we have $d_2(\tilde{H}) = \langle a, b, c\rangle$. In view of $c \geqq 2(a-1)(b-1)-1$ we get $c \geqq c(H)+(a-1)(b-1)-1 \geqq c(H)$, which implies that $c \in \langle a, b\rangle$. Hence, $d_2(\tilde{H}) = \langle a, b\rangle$. By Theorem 2.2 in 2) we see that $\tilde{H}$ is of double covering type. □

Here, we consider some numerical semigroups $\tilde{H}$ satisfying the assumptions in Lemma 3.3, but with $\sharp M(d_2(\tilde{H})) = 3$.

Proposition 4.5. *Let $\tilde{H} = \langle 3, 3\cdot 2n+1, 3\cdot 2n+2\rangle$ with a positive integer n. Then $\tilde{H}$ is of double covering type and toric type. In this case, we have $d_2(\tilde{H}) = \langle 3, 3n+1, 3n+2\rangle$ which is of toric type.*

Proof. By Lemma 3.3 we have

$$d_2(\tilde{H}) = \langle 3, 3\cdot 2n+1, 3n+1, 3n+2\rangle = \langle 3, 3n+1, 3n+2\rangle. \tag{15}$$

We set $H = \langle 3, 3n+1, 3n+2\rangle$, which is Weierstrass. Hence there is a pointed curve (C, P) such that $H(P) = H$. Let K be a canonical divisor on C. Since $3n+2-3 = 3n-1$ is the largest gap at P, we have $K \sim (3n-2)P+E$ where E is an effective divisor with $h^0(E) = 1$ and $h^0(E-P) = 0$. Let Q be a ramification point of $\varphi_{|3P|}$ distinct from P such that $h^0(E-Q) = 0$, because the number of ramification points of $\varphi_{|3P|}$ is larger than or equal to $2n+2 > n+1 = \deg E+1$. We consider a divisor $D = 2P-Q$ on C. Then $2D-P = 3P-2Q \sim Q'$ with $Q' \neq P$. By the Mumford's method we get a double covering $\pi : \tilde{C} \longrightarrow C$. Let $\tilde{P}$ be the ramification point over P. Then we obtain

$$h^0(2\tilde{P}) = h^0(P) + h^0(P-D) = 1 + h^0(-P+Q) = 1 \tag{16}$$

and

$$h^0(4\tilde{P}) = h^0(2P) + h^0(2P-D) = 1 + h^0(Q) = 2, \tag{17}$$

which implies that $3 \in H(\tilde{P})$. Moreover, we have

$$h^0(3\cdot 2n\tilde{P}) = h^0(3nP) + h^0((3n-2)P+Q) = n+1+h^0((3n-2)P+Q) \tag{18}$$

and

$$h^0((3\cdot 2n+2)\tilde{P}) = h^0((3n+1)P) + h^0((3n-1)P+Q) = n+2+h^0((3n-1)P+Q). \tag{19}$$

On the other hand we have

$$h^0((3n-1)P+Q) = 3n+1-2n+h^0(K-(3n-1)P-Q) = n+1+h^0(E-P-Q) = n+1, \tag{20}$$

which implies $h^0((3\cdot 2n+2)\tilde{P}) = 2n+3$. By the assumption on Q we obtain

$$h^0((3n-2)P+Q) = 3n-1+1-2n+h^0(K-(3n-2)P-Q) = n+h^0(E-Q) = n, \tag{21}$$

which implies that $h^0(3\cdot 2n\tilde{P}) = 2n+1$. Thus, we get $3\cdot 2n+1, 3\cdot 2n+2 \in H(\tilde{P})$. Hence, we have $H(\tilde{P}) = \tilde{H}$. □

However, for a numerical semigroup $\tilde{H}$ as in Lemma 3.5 we have the following examples which are not of double covering type, in this case $\sharp M(d_2(\tilde{H})) < 6$.

Example 4.6. Let n be an odd positive integer. Then the numerical semigroup $\tilde{H} = \langle 3, 3n+2, 3(n+1)+1\rangle$ is not of double covering type. In this case we have $d_2(\tilde{H}) = \langle 3, 3\cdot\frac{n+1}{2}+1, 3\cdot\frac{n+1}{2}+2\rangle$. In fact, we have $g(\tilde{H}) = 2n+1$ and $g(d_2(\tilde{H})) = n+1$. Hence, we get $g(\tilde{H}) = 2g(H)-1$. If $\tilde{H}$ were of double covering type, then by Riemann-Hurwitz formula we should have $g(\tilde{H}) > 2g(d_2(\tilde{H}))-1$. Thus, $\tilde{H}$ is not of double covering type.

We will give examples of $\tilde{H}$ with $\sharp M(\tilde{H}) = 4$ which are of double covering type and toric type.

Lemma 4.7. *Let H be a symmetric m-semigroup with $m \neq 2$. Let n be an odd number with $n \geqq 2c(H) - 1$. Then the numerical semigroup $\tilde{H} = 2H + n\mathbb{N}_0$ is symmetric.*

Proof. Let $S(H) = \{m, s_1, \dots, s_{m-1}\}$ be the standard basis for H. We set $s_{max} = \max\{s_1, \dots, s_{m-1}\}$. Since H is symmetric, we have $s_{max} - m + 1 = c(H) = 2g(H)$. By Lemma 2 i) in 2) the standard basis for $\tilde{H}$ consists of $2m, 2s_1, \dots, 2s_{m-1}, n, n + 2s_1, \dots, n + 2s_{m-1}$. Hence, we get $c(\tilde{H}) = n + 2s_{max} - 2m + 1 = n - 1 + 4g(H)$. Moreover, we know from Lemma 2 ii) in 2) that $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2}$, which implies that $c(\tilde{H}) = 2g(\tilde{H})$. Hence, $\tilde{H}$ is symmetric. □

Using Lemma 4.7 we get some desired numerical semigroups generated by four elements.

Proposition 4.8. *Let H be a symmetric semigroup with $\sharp M(H) = 3$ and n an odd number with $n \geqq 2c(H) - 1$. We set $\tilde{H} = 2H + n\mathbb{N}_0$. Then $\tilde{H}$ is of double covering type and toric type.*

Proof. By Lemma 4.7 $\tilde{H}$ is also symmetric. It follows from Proposition 5.2 in 1) that $\tilde{H}$ is of toric type. Moreover, by Theorem 2.2 in 2) $\tilde{H}$ is of double covering type. □

Example 4.9. Let $H = \langle 4, 6, 4m + 1 \rangle$ with a positive integer m. Then $g(H) = 1 + m + m + 1 = 2m + 2$ and $c(H) = 4m + 7 - 4 + 1 = 4m + 4 = 2g(H)$, which implies that H is symmetric. If n is an odd number larger than or equal to $8m + 7$, then $\tilde{H} = \langle 8, 12, 8m + 2, n \rangle$ is of double covering type and toric type.

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