

[研究論文]

On quasi-symmetric numerical semigroups

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Abstract

Symmetric numerical semigroups have good properties, which are given and proved in this paper. The main aim of this paper is to investigate whether the analogies of the properties hold or not when numerical semigroups are quasi-symmetric.

Keywords: Symmetric numerical semigroup, Quasi-symmetric numerical semigroup

1. Introduction

A *numerical semigroup* H is a submonoid of the additive monoid \mathbb{N}_0 of non-negative integers such that its complement $\mathbb{N}_0 \setminus H$ in \mathbb{N}_0 is finite. The cardinality of $\mathbb{N}_0 \setminus H$ is called the *genus* of H , which is denoted by $g(H)$. We set

$$c(H) = \min\{n \in \mathbb{N}_0 \mid n + H \subseteq \mathbb{N}_0\}, \quad (1)$$

which is called the *conductor* of H . It is well-known that $c(H) \leq 2g(H)$ (for example, see 1)). A numerical semigroup H is said to be *symmetric* and *quasi-symmetric* if $c(H) = 2g(H)$ and $c(H) = 2g(H) - 1$ respectively. We define a map d_2 from the set of numerical semigroups into the same set by sending H to the numerical semigroup

$$d_2(H) = \left\{ \frac{h}{2} \mid h \in H \text{ is even} \right\}. \quad (2)$$

For a numerical semigroup H we define a set $p(H) = H \cup \{c(H) - 1\}$, which is a numerical semigroup of genus $g(H) - 1$ (see Bras-Amorós 4)). The semigroup $p(H)$ is called the *parent* of H . Let $\mathbb{N}_0 \setminus H = \{l_1 < l_2 < \cdots < l_{g(H)}\}$. For i with $1 \leq i \leq g(H)$ we set $\alpha_i = l_i - i$. We denote $(\alpha_1, \alpha_2, \dots, \alpha_{g(H)})$ by $\alpha(H)$, which is called the *Schubert index* of H . Then we have $\alpha(p(H)) = (\alpha_1, \dots, \alpha_{g(H)-1})$.

In Section 2 we investigate the properties of symmetric numerical semigroups. It is showed that symmetric numerical semigroups are described by the semigroups $d_2(H)$ using the concept of Torres 2). Using this description we study about the restriction of the map d_2 to the set of symmetric numerical semigroups. Moreover, we characterize the semigroups H such that the semigroups $p(H)$ are symmetric. In Section 3 we are devoted to the study of quasi-symmetric numerical semigroups. We check whether analogies of properties which we get in Section 2 hold or not in these cases.

2. Symmetric numerical semigroups

In this section we will treat only symmetric numerical semigroups. First we state its well-known fundamental property which is the reason that a numerical semigroup H with $c(H) = 2g(H)$ said to be symmetric.

Lemma 2.1. *For a numerical semigroup H the following are equivalent:*

- i) H is symmetric.
- ii) For $\gamma \in \mathbb{Z}$, we have $\gamma \in \mathbb{Z} \setminus H$ if and only if $2g(H) - 1 - \gamma \in H$.

Proof. First, we show that ii) implies i). Since $0 \notin \mathbb{Z} \setminus H$, we get $2g - 1 \notin H$. In view of $c(H) \leq 2g(H)$ we have $c(H) = 2g(H)$.

Assume that H is symmetric, i.e., $2g(H) - 1 \notin H$. We set $g = g(H)$. Let $h \in H$. Assume that $2g - 1 - h \in H$. Then we get $2g - 1 = h + 2g - 1 - h \in H$, which is a contradiction. Hence we have showed that $2g - 1 - \gamma \in H$ implies $\gamma \in \mathbb{Z} \setminus H$. For any integers $a \leq b$ we denote by $[a, b]$ the set of integers n with $a \leq n \leq b$. Then we have a bijection from the set $[0, 2g - 1] \cap H$ to the set $[0, 2g - 1] \cap \mathbb{N}_0 \setminus H$ by sending $2g - 1 - \gamma$ to γ . Thus, $\gamma \in \mathbb{N}_0 \setminus H$ implies $2g - 1 - \gamma \in H$. If $\gamma \in \mathbb{Z}$ with $\gamma < 0$, then $2g - 1 - \gamma \geq 2g - 1 + 1 = 2g$, which implies that $2g - 1 - \gamma \in H$. \square

To describe another characterization of symmetric numerical semigroups we prepare some notation. A numerical semigroup is called an m -semigroup if m is the least positive integer in H . For any integer i with $1 \leq i \leq m-1$ we set $s_i = \min\{h \in H \mid h \equiv i \pmod{m}\}$. The set $S(H) = \{m, s_1, \dots, s_{m-1}\}$ forms a set of generators for the monoid H , which is called the *standard basis* for H . Moreover, we set $s_{\max} = \max\{s_1, \dots, s_{m-1}\}$. We note that $s_{\max} + s_k \notin S(H)$ for all k .

Theorem 2.2. (Komeda 3)) *For a numerical semigroup H the following are equivalent:*

- (i) *If s_i satisfies $s_i + s_k \notin S(H)$ for all k , then $s_i = s_{\max}$.*
- (ii) *H is symmetric.*

When we give numerical semigroups, we use the following notation: For positive integers a_1, \dots, a_n we denote by $\langle a_1, \dots, a_n \rangle$ the monoid generated by a_1, \dots, a_n .

Example 2.3. Let $H = \langle 5, 7, 8, 9 \rangle$. Then we have $S(H) = \{m = 5, s_2 = 7, s_3 = 8, s_4 = 9, s_1 = 16\}$ and $s_{\max} = 16$. Moreover, we obtain $\mathbb{N}_0 \setminus H = \{1, 2, 3, 4, 6, 11\}$. Hence, we get $g(H) = 6$ and $c(H) = 12 = 2g(H)$. Thus, H is symmetric. In this case, $s_2 + s_4 = s_1$ and $s_3 + s_3 = s_1$.

In Torres 2) there is a method of constructing symmetric numerical semigroups for a given numerical semigroups as follows:

Remark 2.4. *Let H' be a numerical semigroup. For $g \geq 4g(H')$ the set*

$$H = 2H' \cup \{2g - 1 - 2t \mid t \in \mathbb{Z} \setminus H'\}$$

is a symmetric numerical semigroup of genus g with $d_2(H) = H'$ where we set $2H' = \{2h' \mid h' \in H'\}$.

Any symmetric numerical semigroup can be described as in the above.

Proposition 2.5. *If H is a symmetric numerical semigroup, then we have*

$$H = 2d_2(H) \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z} \setminus d_2(H)\}. \quad (3)$$

Proof. We have $\{h \in H \mid h \text{ is even}\} = 2d_2(H)$. We set $g = g(H)$. It suffices to show that

$$\{h \in H \mid h \text{ is odd}\} = \{2g(H) - 1 - 2t \mid t \in \mathbb{Z} \setminus d_2(H)\}. \quad (4)$$

Let $t \in \mathbb{Z} \setminus d_2(H)$. Then we have $2t \notin H$. Since H is symmetric, by Lemma 2.1 we get $2g - 1 - 2t \in H$. Conversely, let $h \in H$ be odd. Since $h = 2g - 1 - (2g - 1 - h) \in H$, we get $2g - 1 - h \notin H$, which is even. We set $2g - 1 - h = 2t \notin H$. Then $t \notin d_2(H)$. Hence, we obtain $h = 2g - 1 - 2t$ with $t \in \mathbb{Z} \setminus d_2(H)$. \square

Hence, we can give a description of a symmetric numerical semigroup as follows:

Theorem 2.6. *Let H be a numerical semigroup. Then the following are equivalent:*

- (i) *H is symmetric.*
- (ii) *$H = 2d_2(H) \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z} \setminus d_2(H)\}$.*

Example 2.7. Let $H = \langle 5, 7, 8, 9 \rangle$, whose genus is 6. It is symmetric. We have $d_2(H) = \langle 4, 5, 6, 7 \rangle$. We obtain

$$H = 2\langle 4, 5, 6, 7 \rangle \cup \{11 - 2t \mid t \in \mathbb{Z} \setminus \langle 4, 5, 6, 7 \rangle\} = \{n \in \mathbb{Z} \mid n < 0\} \cup \{1, 2, 3\}. \quad (5)$$

Let $d_{2,s}$ be the restriction map of d_2 to the set of symmetric numerical semigroups and $d_{2,s,g}$ the restriction map of d_2 to the set of symmetric numerical semigroups of genus g . Then by Remark 2.4, Proposition 2.5 and Theorem 2.6 these maps have the following properties:

Theorem 2.8. (i) *The map $d_{2,s}$ is surjective.*

(ii) *For any numerical semigroup H' we have $\sharp d_{2,s}^{-1}(H') = \infty$.*

(iii) *For a fixed non-negative integer g the map $d_{2,s,g}$ is injective.*

(iv) *Let g be any non-negative integer and H' any numerical semigroup. Then we have $\sharp d_{2,s,g}^{-1}(H') \leq 1$. Moreover, if $g \geq 4g(H')$, then we get $\sharp d_{2,s,g}^{-1}(H') = 1$.*

Finally, we want to characterize the numerical semigroups H whose parents $p(H)$ are symmetric.

Theorem 2.9. *For a numerical semigroup H of genus $g \geq 1$ the following are equivalent:*

- (i) $p(H)$ is symmetric.
- (ii) H is either $\langle 2, 2g + 1 \rangle$ or $\langle 3, 4, 5 \rangle$ whose genus is 2.

Proof. Let $H = \langle 2, 2g + 1 \rangle$. Then $p(H) = \langle 2, 2g - 1 \rangle$, which is symmetric. Let $H = \langle 3, 4, 5 \rangle$. Then $p(H) = \langle 2, 3 \rangle$, which is symmetric.

Assume that $p(H)$ is symmetric. Then we have the Schubert index $\alpha(p(H)) = (\alpha_1, \dots, \alpha_{g-2}, g-2)$. Hence, we get $\alpha(H) = (\alpha_1, \dots, \alpha_{g-2}, g-2, \alpha_g)$ where $\alpha_g = g-2$ or $g-1$. First, we consider the case $\alpha_g = g-1$. Then H is symmetric. Moreover, we get $g-2 + g-1 = 2g-3 \notin H$. Hence, by Lemma 2.1 we obtain $H \ni 2g-1 - (2g-3) = 2$, which implies that $H = \langle 2, 2g + 1 \rangle$. Next, let $\alpha_g = g-2$. Then we have $g + g-2 = 2g-2 \notin H$ and $2g-1 \in H$, which implies that H is quasi-symmetric. Moreover, we get $H \ni g-2 + g-1 = 2g-3$. If $g \neq 2$, then we get $2g-3 \neq g-1$. By Lemma 3.1 in the next Section we obtain $H \ni 2g-2 - (2g-3) = 1$, which implies that $H = \mathbb{N}_0$. This is a contradiction. If $g = 2$, then we have $\alpha(H) = (0, 0)$, which means $H = \langle 3, 4, 5 \rangle$. \square

3. Quasi-symmetric numerical semigroups

First, we give a characterization of a quasi-symmetric semigroup which is analogous to Lemma 2.1.

Lemma 3.1. *Let H be a numerical semigroup of genus g . Then the following are equivalent:*

- (i) H is quasi-symmetric.
- (ii) Let $\gamma \neq g-1$, Then $\gamma \in \mathbb{Z} \setminus H$ if and only if $2g-2-\gamma \in H$.

Proof. Assume that (ii) holds. Let $\gamma = -1$. Then $\gamma \in \mathbb{Z} \setminus H$. Hence, $H \ni 2g-2-\gamma = 2g-1$. Let $\gamma = 2g-2$. Then $2g-2-\gamma = 0 \in H$. Hence, $\mathbb{Z} \setminus H \ni \gamma = 2g-2$, which implies that $c(H) = 2g-1$.

Next, we prove that (i) implies (ii). Let us consider the bijective map

$$[0, 2g-2] \setminus \{g-1\} \longrightarrow [0, 2g-2] \setminus \{g-1\}$$

sending n to $2g-2-n$. Since H is quasi-symmetric, $2g-2 \notin H$. Hence, we have

$$(\mathbb{N}_0 \setminus H) \setminus \{g-1\} = \{2g-2-h \mid h \in H \cap [0, 2g-2] \setminus \{g-1\}\}. \quad (6)$$

We may assume that $\gamma > 0$, because $c(H) = 2g-1$. Let $\gamma \in \mathbb{N}_0 \setminus H$ with $\gamma \neq g-1$. Then $\gamma = 2g-2-h$ with $h \in H \cap [0, 2g-2] \setminus \{g-1\}$. Hence, $2g-2-\gamma = h \in H$. Conversely, let $2g-2-\gamma \in H$ with $\gamma \neq g-1$. Then $\gamma = 2g-2-(2g-2-\gamma) \in (\mathbb{N}_0 \setminus H) \setminus \{g-1\}$. \square

By Lemma 3.1 we have the following description of the standard basis in the case where H is quasi-symmetric.

Proposition 3.2. *If H is a quasi-symmetric m -semigroup of genus g , then we have the following:*

if $i \in \{1, 2, \dots, m-1\}$ satisfies $s_i + s_k \notin S(H)$ for all k , then $s_i = s_{max}$ or $s_i = g-1+m$.

Proof. Assume that $s_i \neq s_{max}$ and $s_i \neq g-1+m$. By Lemma 3.1 we have

$$s_{max} = 2g-2+m = 2g-2-(s_i-m) + s_i = s_k + s_i \quad (7)$$

for some k . \square

Example 3.3. Let $H = \langle 5, 9, 12, 13 \rangle$. Then we have $S(H) = \{m = 5, s_4 = 9, s_2 = 12, s_3 = 13, s_1 = 21\}$ and $s_{max} = 21$. Moreover, we obtain $\mathbb{N}_0 \setminus H = \{1, 2, 3, 4, 6, 7, 8, 11, 16\}$. Hence, we get $g(H) = 9$ and $c(H) = 17 = 2c(H) - 1$. Thus, H is quasi-symmetric. In this case, we have $s_4 + s_2 = s_1$ and $g(H) - 1 + m = 13 = s_3$.

By Proposition 3.2 if H is quasi-symmetric, then we get $\#\{i \mid s_i + s_k \notin S(H) \text{ for all } k\} = 2$. But its converse does not hold. In fact, we have the following examples:

Example 3.4. Let $m \geq 3$ and $\nu \geq 1$ with $(m, \nu) \neq (3, 1)$. We set $H_{m, \nu} = \langle m, \nu m + 1, (m-2)\nu m + m - 1 \rangle$. Then we have

$$\#\{i \mid s_i + s_k \notin S(H) \text{ for all } k\} = 2, \quad (8)$$

but $H_{m,\nu}$ is not quasi-symmetric.

Proof. We have $s_i = i\nu m + i$ for $1 \leq i \leq m-2$ and $s_{m-1} = (m-2)\nu m + m-1$. In this case, $s_{max} = s_{m-1}$. Hence we have

$$g(H_{m,\nu}) = \nu \sum_{i=1}^{m-2} i + \nu(m-2) = \nu \cdot \frac{(m-2)(m+1)}{2}. \quad (9)$$

Moreover, $c(H_{m,\nu}) = s_{max} - m + 1 = (m-2)\nu m$. Thus, we obtain

$$2g(H_{m,\nu}) - 1 - c(H_{m,\nu}) = \nu(m-2)(m+1) - 1 - \nu(m-2)m = \nu(m-2) - 1. \quad (10)$$

By the assumption on m and ν we have $2g(H_{m,\nu}) - 1 - c(H_{m,\nu}) > 0$, which implies that $H_{m,\nu}$ is neither quasi-symmetric nor symmetric. On the other hand, for any $1 \leq i \leq m-3$ we have

$$s_i + s_{m-2-i} = i\nu m + i + (m-2-i)\nu m + m-2-i = s_{m-2}. \quad (11)$$

Moreover, we have

$$s_{m-2} + s_1 = (m-1)\nu m + m-1 > s_{max}. \quad (12)$$

Hence, we get (8). \square

Let $d_{2,qs}$ be the restriction map of d_2 to the set of quasi-symmetric numerical semigroups and $d_{2,qs,g}$ the restriction map of d_2 to the set of quasi-symmetric numerical semigroups of genus g . The analogy to Theorem 2.8 (i) does not hold in the quasi-symmetric case.

Remark 3.5. The map $d_{2,qs}$ is not surjective.

Proof. We give a counterexample. For example, let H be a numerical semigroup with $d_2(H) = \mathbb{N}_0$. Then H must contain 2, which implies $H = \langle 2, 2g+1 \rangle$ for some $g \in \mathbb{N}_0$. Thus, H is symmetric. Hence, we have $d_{2,qs}^{-1}(\mathbb{N}_0) = \emptyset$. \square

To investigate the fibers of the maps $d_{2,qs}$ and $d_{2,qs,g}$ we need the following lemma:

Lemma 3.6. *Let H be a quasi-symmetric numerical semigroup. Then we have $g(H) - 1 \notin d_2(H)$.*

Proof. Since H is quasi-symmetric, we have $2g(H) - 2 \notin H$. Hence, we get $g(H) - 1 \notin d_2(H)$. \square

By the following we know that the analogy of Theorem 2.8 (ii) does not hold in our case.

Proposition 3.7. *For any numerical semigroup H' we have $d_{2,qs}^{-1}(H') < \infty$.*

Proof. Let H be a quasi-symmetric numerical semigroup such that $d_2(H) = H'$. By Lemma 3.6 we have $g(H) - 1 \notin H'$. Hence, we get

$$d_{2,qs}^{-1}(H') \subseteq \cup_{\gamma \in \mathbb{N}_0 \setminus H'} \{\text{numerical semigroups of genus } \gamma + 1\}. \quad (13)$$

Thus, we get $d_{2,qs}^{-1}(H') < \infty$. \square

Moreover, the analogy of Theorem 2.8 (iii) fails.

Proposition 3.8. *There exist a non-negative integer g and a numerical semigroup H' such that $\sharp d_{2,qs,g}^{-1}(H') \geq 2$.*

Proof. We will give a non-negative integer g and a numerical semigroup H' such that $\sharp d_{2,qs,g}^{-1}(H') = 3$. Let $H' = \langle 5, 6, 7, 9 \rangle$. Then $\mathbb{N}_0 \setminus H' = \{1, 2, 3, 4, 8\}$, hence $g(H') = 5$ and $c(H') = 9$. Let H be a quasi-symmetric numerical semigroup with $d_2(H) = H'$. By Lemma 3.6 we have $g(H) = \gamma + 1$ and $\gamma \geq 5$. Hence, we get $\gamma = 8$ and $g(H) = 9$. We set

$$H_1 = \langle 5, 7, 13 \rangle, H_2 = \langle 5, 9, 12, 13 \rangle, H_3 = \langle 7, 10, 11, 12, 13, 15 \rangle \text{ and } H_4 = \langle 9, 10, 11, 12, 13, 14, 15, 17 \rangle. \quad (14)$$

These four semigroups are quasi-symmetric. Moreover, we have $d_{2,qs}^{-1}(H') = d_{2,qs,g}^{-1}(H') = \{H_1, H_2, H_3, H_4\}$. \square

We determine the numerical semigroups whose parents are quasi-symmetric, like in the case where the parents are symmetric.

Theorem 3.9. *For a numerical semigroup H of genus $g \geq 3$ the following are equivalent:*

- (i) $p(H)$ is quasi-symmetric.
- (ii) H is either $\langle 3, g+1 \rangle$ with $g \not\equiv 2 \pmod 3$ or $\langle 3, 5, 7 \rangle$ or a numerical semigroup with Schubert index $(0^{g-2}, g-3, g-3)$.

Proof. First, we note that a numerical semigroup of genus $g \leq 1$ is either \mathbb{N}_0 or $\langle 2, 3 \rangle$, which is symmetric. Hence, we assume that $g = g(H) \geq 3$.

Let $H = \langle 3, g+1 \rangle$ with $g \not\equiv 2 \pmod 3$. Then the last gap of H is $2(g+1)-3 = 2g-1$. Hence, we get $p(H) = \langle 3, g+1, 2g-1 \rangle$, whose last gap is $2g-1-3$. Thus, we have $c(p(H)) = 2g-3 = 2(g-1)-1 = 2g(p(H))-1$, which implies that $p(H)$ is quasi-symmetric.

Let $H = \langle 3, 5, 7 \rangle$, whose last gap is 4. Hence, we get $p(H) = \langle 3, 4, 5 \rangle$, which is quasi-symmetric.

Let H be a numerical semigroup with $\alpha(H) = (0^{g-2}, g-3, g-3)$. Then we have $\alpha(p(H)) = (0^{g-2}, g-3)$. Hence, the last gap of $p(H)$ is $g-3+g-1 = 2(g-1)-2$, which implies that it is quasi-symmetric.

Conversely, we assume that $p(H)$ is quasi-symmetric. Then its Schubert index $\alpha(p(H))$ is $(\alpha_1, \dots, \alpha_{g-2}, g-3)$. Hence, we get $\alpha(H) = (\alpha_1, \dots, \alpha_{g-2}, g-3, \alpha_g)$ where α_g is either $g-1$ or $g-2$ or $g-3$.

Let $\alpha_g = g-1$. Then H is symmetric. Since $H \not\ni g-3+g-1 = 2g-4$, we obtain $H \ni 2g-1-(2g-4) = 3$. Hence, we have $H = \langle 3, n \rangle$ with $n \not\equiv 0 \pmod 3$. Here, we note that any 3-semigroup generated by three elements which is not generated by two elements is not symmetric. In view of $g = g(H) = \frac{(3-1)(n-1)}{2}$, we get $H = \langle 3, g+1 \rangle$ with $g \not\equiv 2 \pmod 3$,

Let $\alpha_g = g-2$. Then H is quasi-symmetric. We have $2g-4 \notin H$. Let $g \neq 3$. Then we get $2g-4 \neq g-1$, which implies that $H \ni 2g-2-(2g-4) = 2$ by Lemma 3.1. Hence, we should have $H = \langle 2, 2g+1 \rangle$, which is symmetric. This is a contradiction. If $g = 3$, then we have $\alpha(H) = (0, 0, 1)$. Hence, we get $H = \langle 3, 5, 7 \rangle$.

Let $\alpha_g = g-3$. We have $\alpha(H) = (\alpha_1, \dots, \alpha_{g-2}, g-3, g-3)$. Assume that $\alpha_{g-2} \neq 0$. Then we have $H \not\ni \alpha_{g-2} + g-2$, which is distinct from $g-2$. We note that $2g-1, 2g-2 \in H$ and $2g-3, 2g-4 \notin H$. Hence, we have a bijection Φ between the sets $H \cap [0, 2g-4]$ and $(\mathbb{N}_0 \setminus H \cap [0, 2g-4]) \setminus \{g-2\}$ sending h to $2g-4-h$. Since we have $\Phi^{-1}(g-2-\alpha_{g-2}) = g-2+\alpha_{g-2}$, the integer $g-2-\alpha_{g-2}$ belongs to H . Hence, we obtain $g-1+\alpha_{g-2} \notin H$, because we have $2g-3 = (g-2-\alpha_{g-2}) + (g-1+\alpha_{g-2})$ with $2g-3 \notin H$. Since $g-1+\alpha_{g-2} \notin H$ is larger than $g-2+\alpha_{g-2}$, we get $g-1+\alpha_{g-2} = 2g-4$ or $2g-3$. If $g-1+\alpha_{g-2} = 2g-3$, then we have $g-3 \geq \alpha_{g-2} = g-2$, which is a contradiction. Hence, we may assume that $g-1+\alpha_{g-2} = 2g-4$, i.e., $\alpha_{g-2} = g-3$. Therefore, we get $H \not\ni \alpha_{g-2} + g-2 = g-3+g-2 = 2g-5$. Since we have $\Phi^{-1}(2g-5) = 1$, we obtain $1 \in H$, which implies that $H = \mathbb{N}_0$. This is a contradiction. Thus, we get $\alpha(H) = (0^{g-2}, g-3, g-3)$. \square

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