[研究論文] # Symmetric Weierstrass numerical semigroups whose quotients by two are also symmetric and Weierstrass

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Abstract

We investigate whether for any g and g' there are symmetric Weierstrass numerical semigroups H of genus g such that the numerical semigroups H' of genus g' consisting of the elements $h/2$ with even h in H are also symmetric and Weierstrass. In fact, there are no such pairs (H, H') for $g = 3g' - 1$ or $g \leqq 2g' - 1$.

Keywords: Symmetric numerical semigroup, Weierstrass numerical semigroup

1. Introduction

A *numerical semigroup* H is a submonoid of the additive monoid $\mathbb{N}_0$ of non-negative integers such that its complement $\mathbb{N}_0 \backslash H$ in $\mathbb{N}_0$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. We set

$$c(H) = \min\{n \in \mathbb{N}_0 \mid n + \mathbb{N}_0 \subseteq H\},$$

which is called the *conductor* of H. It is well-known that $c(H) \leqq 2g(H)$. A numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$. We define a map d_2 from the set of numerical semigroups into the same set by sending H to the numerical semigroup

$$d_2(H) = \left\{ \frac{h}{2} \,\middle|\, h \in H \text{ is even} \right\}.$$

Let C be a complete nonsingular irreducible curve over an algebraically closed field k of characterisic 0, which is called a *curve* in this paper. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\}$$

where $k(C)$ is the field of rational functions on C and $(f)_\infty$ means the polar divisor of f. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. Buchweitz first found a non-Weierstrass numerical semigroup in 1980 (See 1)). So the following problem is posed:

Problem. *Find a necessary and sufficient computable condition for a numerical semigroup to be Weierstrass, which we call Hurwitz' problem.*

We showed the following in 2):

Fact 1. *Let H be a symmetric numerical semigroup with $g(H) \geqq 4g(d_2(H))$. If $d_2(H)$ is Weieerstrass, then so is H.*

Combining Fact 1 with the result of Torres (see 3)) we get the following:

Fact 2. *Let H be a symmetric numerical semigroup with $g(H) \geqq 6g(d_2(H)) + 4$. Then $d_2(H)$ is Weierstrass if and only if so is H.*

From Fact 2 we remove the condition $g(H) \geqq 6g(d_2(H)) + 4$ and assume that $d_2(H)$ is symmetric. But in this case it is difficult to solve the problem corresponding to Fact 2. So in this paper we only consider whether for any g and g' we can construct symmetric Weierstrass numerical semigroups H of genus g with symmetric and Weierstrass $d_2(H)$ of genus g'.

2. Case $g(H) \geqq 4g(d_2(H))$

For any numerical semigroup H' and any $g \geqq 4g(H')$ we can construct a symmetric numerical semigroup H of genus g as follows (see 3)):

$$H = 2H' \cup \{2g-1-2t \mid t \in \mathbb{Z}\backslash H'\},$$

which is denoted by $S(H', g)$. Then we have $d_2(S(H', g)) = H'$. Hence, by Fact 1 in Introduction, if H' is Weierstrass, then so is H.

3. Case $3g(d_2(H)) \leqq g(H) \leqq 4g(d_2(H)) - 1$

To describe a numerical semigroup H we use the following notations: We set

$$m(H) = \min\{h \in H \mid h > 0\} \text{ and } S(H) = \{m(H), s_1, \ldots, s_{m(H)-1}\}$$

where $s_i = \min\{h \in H \mid h \equiv i \mod m(H)\}$ for $i = 1, \ldots, m(H) - 1$. We call $S(H)$ the *standard basis* for H. For any positive integers $a_1, a_2, \ldots, a_n$ we denote by $\langle a_1, a_2, \ldots, a_n\rangle$ the additive monoid generated by $a_1, a_2, \ldots, a_n$. Using the following lemma we will give symmetric Weierstrass numerical semigroups H with symmetric and Weierstrass $d_2(H)$ in our case.

Lemma 3.1. *Let H' be a numerical semigroup distinct from $\mathbb{N}_0$ and n an odd number $\geqq c(H') + m(H') - 1$. Then we have the following:*
i) $g(2H' + n\mathbb{N}_0) = 2g(H') + \dfrac{n-1}{2}$.
ii) *If H' is symmetric, then so is $2H' + n\mathbb{N}_0$. In this case if H' is Weierstrass, then so is $2H' + n\mathbb{N}_0$.*

Proof. In the proof of Lemma 2.1 in 4) we may replace the assumption $n \geqq 2c(H') - 1$ by $n \geqq c(H') + m(H') - 1$. Hence, we get i).

We have $c(2H' + n\mathbb{N}_0) = n + 2s_{max} - 2m(H') + 1$ where we set $s_{max} = \max\{s_i \mid i = 1, \ldots, m(H') - 1\}$. Since we have $s_{max} - m(H') + 1 = c(H') = 2g(H')$, we get

$$c(2H' + n\mathbb{N}_0) = n - 1 + 4g(H') = 2g(2H' + n\mathbb{N}_0)$$

by i), which implies that $2H' + n\mathbb{N}_0$ is symmetric. Since H' is a symmetric numerical semigroup distinct from $\mathbb{N}_0$, we obtain $c(H') + m(H') - 1 \geqq 2g(H') + 1$. The inequality $\deg(2D - P) \geqq 2r + 1 = 2g(H') + 1$ in the proof of Theorem 2.2 of 4) holds even if the assumption $n \geqq 2c(H') - 1$ is replaced by $n \geqq \max\{c(H') + m(H') - 1, 2g(H') + 1\} = c(H') + m(H') - 1$. Moreover, using Remark 3.2 of 3) the equality $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(n\tilde{P})) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n-1)\tilde{P})) + 1$ in the proof of Theorem 2.2 of 4) holds with no assumption on n. Here, we use the notations in the proof of Theorem 2.2 of 4). Hence, if H' is Weierstrass, then so is $2H' + n\mathbb{N}_0$. □

In our case we get the following:

Proposition 3.2. *Let s and r be non-negative integers with $r \geqq 1$. We set $H' = \langle 2(s+1), 2(s+1)r + 1\rangle$. Then for any integer $t \geqq s$ there exists a symmetric Weierstrass numerical semigroup H of genus $3g(H') + t$ with $d_2(H) = H'$.*

Proof. We have

$$c(H') + m(H') - 1 = 2g(H') + 2(s+1) - 1 = 2(g(H') + s) + 1.$$

If we set $n = 2(g(H') + t) + 1$ with an integer $t \geqq s$, then the numerical semigroup $H = 2H' + n\mathbb{N}_0$ with $d_2(H) = H'$ is symmetric and Weierstrass by Lemma 3.1 , because every numerical semigroup generated by two elements is Weierstrass (for example, see Corollary 2. 4 in 5)). In this case we obtain

$$g(2H' + n\mathbb{N}_0) = 2g(H') + \frac{n-1}{2} = 3g(H') + t.$$

□

4. Case $2g(d_2(H)) \leqq g(H) \leqq 3g(d_2(H)) - 1$

We have the following symmetric Weierstrass numerical semigroups H' and H with $d_2(H) = H'$ from which we will get the desired pairs of symmetric Weierstrass numerical semigroups.

Lemma 4.1. *Let p be an odd number and q an integer with $1 \leqq q \leqq p-1$ and $(p,q)=1$. Moreover, let l and r be positive integers. We set $H' = \langle p, pl+q\rangle$ and $H = 2H' + (2pr+p)\mathbb{N}_0$. Then we have*

$$g(H) = 2g(H') + pr + \frac{p-1}{2} = 3g(H') - \left(\left(\frac{(p-1)l}{2} - r\right)p + \frac{p-1}{2}(q-2)\right).$$

Moreover, H is symmetric and Weierstrass.

Proof. We have

$$S(H) = \{2p, 2s_1, \ldots, 2s_{p-1}, 2pr+p, 2pr+p+2s_1, \ldots, 2pr+p+2s_{p-1}\}$$

with $s_i = i(pl+q)$ for any $i = 1, \ldots, p-1$, because for any integer j the number $j(2pr+p)$ is congruent to 0 or p modulo $2p$. Hence, we get

$$\begin{aligned} g(H) &= \sum_{i=1}^{p-1}\left\lfloor \frac{2s_i}{2p}\right\rfloor + \left\lfloor \frac{2pr+p}{2p}\right\rfloor + \sum_{i=1}^{p-1}\left\lfloor \frac{2pr+p+2s_i}{2p}\right\rfloor \\ &= g(H') + r + \sum_{i=1}^{p-1}\left(r + \left\lfloor \frac{2s_i+p}{2p}\right\rfloor\right) \\ &= g(H') + r + (p-1)r + g(H') + \frac{p-1}{2} = 2g(H') + pr + \frac{p-1}{2} \\ &= 3g(H') - \left(\frac{(p-1)(pl+q-1)}{2} - pr - \frac{p-1}{2}\right) \\ &= 3g(H') - \left(\left(\frac{(p-1)l}{2} - r\right)p + \frac{p-1}{2}(q-2)\right). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} c(H) &= 2pr + p + 2s_{p-1} - 2p + 1 = 2(s_{p-1} - p + 1) - 1 + 2pr + p \\ &= 2c(H') + 2\left(pr + \frac{p-1}{2}\right) = 2\left(2g(H') + pr + \frac{p-1}{2}\right) = 2g(H), \end{aligned}$$

which implies that H is symmetric. Since H is generated by three elements, it is Weierstrass (for example, see Corollary 2.4 and Proposition 2.5 in 5)). □

Proposition 4.2. *There is no pair (H, H') of symmetric numerical semigroups with $d_2(H) = H'$ and $g(H) = 3g(H') - 1$.*

Proof. Assume that there were a pair (H, H') of symmetric numerical semigroups with $d_2(H) = H'$ and $g(H) = 3g(H') - 1$. By Proposition 2.5 in 6) we obtain

$$H = 2H' \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z}\backslash H'\}.$$

Since H' is symmetric, we have $2g(H') - 1 = c(H') - 1 \notin H'$. Hence, we get

$$H \ni 2g(H) - 1 - 2(2g(H') - 1) = 6g(H') - 2 - 1 - 4g(H') + 2 = 2g(H') - 1,$$

which implies that $4g(H') - 2 \in H$. Thus, we obtain $2g(H') - 1 \in H'$. This is a contradiction. □

But we get the following:

Proposition 4.3. *For any integer $i \geqq 2$ there exists a pair (H, H') of symmetric Weierstrass numerical semigroups with $d_2(H) = H'$ and $g(H) = 3g(H') - i$.*

Proof. In Lemma 4.1 let $r = \dfrac{(p-1)l}{2} - 1$ and $q = 1$. Namely we consider

$$H' = \langle p, pl+1\rangle \text{ and } H = 2H' + \left(2p\left(\frac{(p-1)l}{2} - 1\right) + p\right)\mathbb{N}_0$$

where p is an odd number and l is a positive integer. By Lemma 4.1 we get

$$g(H) = 3g(H') - \left(p - \frac{p-1}{2}\right) = 3g(H') - \frac{p+1}{2}.$$

For any $i \geqq 2$ we set $p = 2i - 1$. Then we get the desired pair (H, H') of symmetric Weierstrass numerical semigroups. □

Next, we construct pairs (H, H') of symmetric Weierstrass numerical semigroups when we view $g(H)$ from the range near $2g(H')$.

Proposition 4.4. *For any integer $i \geqq 1$ there exists a pair (H, H') of symmetric Weierstrass numerical semigroups with $d_2(H) = H'$ and $g(H) = 2g(H') + i$.*

Proof. We set $H' = \langle a, b\rangle$ with $3 \leqq a < b$ and $(a, b) = 1$ where a is odd. Let $H = \langle a, 2b\rangle$. Then we get $d_2(H) = H'$. In fact, we have $d_2(H) \supseteqq \langle a, b\rangle$. Assume that there were $x \in d_2(H)\backslash\langle a, b\rangle$. Then $2x = na + 2mb$ for some non-negative integers n and m. Since a is odd, n must be even. Thus, we obtain $x = \frac{n}{2} \cdot a + mb \in \langle a, b\rangle$. This is a contradiction. We have $g(H') = \frac{(a-1)(b-1)}{2}$. Hence, we obtain

$$g(H) = \frac{(a-1)(2b-1)}{2} = \frac{(a-1)(2b-2) + (a-1)}{2} = 2g(H') + \frac{a-1}{2}.$$

We note that H and H' are symmetric, because they are generated by two elements. For any $i \geqq 1$ if we set $a = 2i + 1$, then we get the desired pairs (H, H'). □

Finally, we investigate the case where $g(H) = 2g(H')$.

Proposition 4.5. *If (H, H') is a pair of symmetric numerical semigroups with $d_2(H) = H'$ and $g(H) = 2g(H')$, then we have $H = H' = \mathbb{N}_0$.*

Proof. By Proposition 2.5 in 6) we obtain

$$H = 2H' \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z}\backslash H'\}.$$

Since H' is symmetric, we have $2g(H') - 1 = c(H') - 1 \notin H'$. Hence, we get

$$H \ni 2g(H) - 1 - 2(2g(H') - 1) = 4g(H') - 1 - 4g(H') + 2 = 1,$$

which implies that $H = \mathbb{N}_0$ and $H' = d_2(H) = \mathbb{N}_0$. □

5. Case $g(H) \leqq 2g(d_2(H)) - 1$

In the final case we get the following using Proposition 2.5 in 6):

Proposition 5.1. *There is no pair (H, H') of symmetric numerical semigroups with $d_2(H) = H'$ and $g(H) \leqq 2g(H') - 1$.*

Proof. We have

$$H = 2H' \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z}\backslash H'\}.$$

Since H' is symmetric, we have $2g(H') - 1 = c(H') - 1 \notin H'$. Hence, we get

$$H \ni 2g(H) - 1 - 2(2g(H') - 1) \leqq 2(2g(H') - 1) - 1 - 4g(H') + 2 = -1,$$

which is a contradiction. □

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