

[研究論文] Diagrams of Buchweitz numerical semigroups

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Abstract

We investigate Buchweitz numerical semigroups H whose quotient $d_2(H)$ by two are also Buchweitz and which satisfy that the numerical semigroups $p(H)$ whose genera decrease one from H are also Buchweitz. Moreover, we give examples of Buchweitz numerical semigroups H such that both $d_2(H)$ and $p(H)$ are not Buchweitz.

Keywords: Buchweitz numerical semigroup, Non-Weierstrass numerical semigroup

1. Introduction

A *numerical semigroup* H is a submonoid of the additive monoid \mathbb{N}_0 of non-negative integers such that its complement $\mathbb{N}_0 \setminus H$ in \mathbb{N}_0 is finite. The cardinality of $L(H) = \mathbb{N}_0 \setminus H$ is called the *genus* of H , which is denoted by $g(H)$. For a positive integer $m \geq 2$ we set

$$L_m(H) = \{l_1 + \cdots + l_m \mid l_i \in L(H), \text{ all } i\}.$$

A numerical semigroup H is said to be *Buchweitz* if there exists an integer $m \geq 2$ such that $\#L_m(H) > (2m - 1)(g(H) - 1)$. Let C be a complete nonsingular irreducible curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\} \quad (1)$$

where $k(C)$ is the field of rational functions on C . We call $H(P)$ the *Weierstrass semigroup* of P . A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. Buchweitz first found a non-Weierstrass numerical semigroup in 1980 (See 1)). In fact, he showed that every Buchweitz numerical semigroup is non-Weierstrass. We set

$$c(H) = \min\{n \in \mathbb{N}_0 \mid n + \mathbb{N}_0 \subseteq H\},$$

which is called the *conductor* of H . It is well-known that $c(H) \leq 2g(H)$. We define a map d_2 from the set \mathcal{H} of numerical semigroups into the same set \mathcal{H} by sending H to the numerical semigroup

$$d_2(H) = \left\{ \frac{h}{2} \mid h \in H \text{ is even} \right\}.$$

We denote by p the map from \mathcal{H} to itself sending H to $p(H) = H \cup \{c(H) - 1\}$. Then we have $g(p(H)) = g(H) - 1$. Recently Kaplan and Ye²⁾ proved that the set of Buchweitz numerical semigroups has the density zero in the set \mathcal{H} of all numerical semigroups. So, Buchweitz numerical semigroups are special in the entire set \mathcal{H} . Moreover, most numerical semigroups of lower genus are Weierstrass. In fact, every numerical semigroups of genus $g \leq 8$ is Weierstrass^{3),4)}. For that reason it is important to investigate Buchweitz numerical semigroups H such that $d_2(H)$ and $p(H)$ are also Buchweitz. In this paper we give such examples. Moreover, we also construct Buchweitz numerical semigroups H such that both $d_2(H)$ and $p(H)$ are not Buchweitz.

2. Buchweitz diagrams of numerical semigroups

Let H be a numerical semigroup and $\gamma(H)$ the largest integer which dose not belong to H . Let m the multiplicity of H , i.e. the least positive integer in H . Such a numerical semigroup H is called an *m-semigroup*. An *m-semigroup* said to be *primitive* if $\gamma(H) < 2m$. When two integers a and b satisfy $a < b$, the symbol $a \rightarrow b$ means the consecutive integers $a, a + 1, \dots, b$. First we review the method to construct the primitive *m-semigroups*.

Proposition 2.1. *Let m be an integer ≥ 2 . Then we have a bijective correspondence between the set of the subsets of $\{m+1 \rightarrow 2m-1\}$ and the set of the primitive m -semigroups sending S to H_S with $\mathbb{N}_0 \setminus H_S = \{1 \rightarrow m-1\} \cup S$.*

Proof. It suffices to check that H_S becomes an additive monoid. If h_1 and h_2 are positive integers in H_S , then we have $2m \leq h_1 + h_2$ which dose not belong to $\mathbb{N}_0 \setminus H_S$. Hence we get $h_1 + h_2 \in H_S$. \square

A diagram of numerical semigroups

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) & & \end{array}$$

is said to be *Buchweitz* if H , $d_2(H)$ and $p(H)$ are Buchweitz.

To describe a numerical semigroup we use the following notation: For any non-negative integers a_1, a_2, \dots, a_n we denote by $\langle a_1, a_2, \dots, a_n \rangle$ the additive monoid generated by a_1, a_2, \dots, a_n .

Example 2.1. Let H be a primitive 26-semigroup with

$$\mathbb{N}_0 \setminus H = \{1 \rightarrow 25\} \cup \{38, 42, 43, 48, 49, 50, 51\}.$$

Then we have

$$\mathbb{N}_0 \setminus d_2(H) = \{1 \rightarrow 12\} \cup \{19, 21, 24, 25\} \text{ and } \mathbb{N}_0 \setminus p(H) = \{1 \rightarrow 25\} \cup \{38, 42, 43, 48, 49, 50\}.$$

The numerical semigroup

$$d_2(H) = \langle 13 \rightarrow 18, 20, 22, 23 \rangle$$

is a famous Buchweitz numerical semigroup which Buchweitz gave in his paper 1). We will prove that

$$\sharp L_2(H) = 95 = 3g(H) - 1.$$

We set $A_2(H) = \{a + a' \mid a, a' \in A(H)\}$ where

$$A(H) = \{l - 1 \mid l \in \mathbb{N}_0 \setminus H\} = \{0 \rightarrow 24\} \cup \{37, 41, 42, 47, 48, 49, 50\}.$$

Then we have

$$A_2(H) = \{0 \rightarrow 74\} \cup \{78, 79, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 94, 95, 96, 97, 98, 99, 100\}.$$

Hence, we get

$$\sharp L_2(H) = 75 + 20 = 95 = 3 \times 32 - 1 = 3g(H) - 1.$$

Moreover, we obtain

$$A_2(p(H)) = \{l_1 + l_2 - 2 \mid l_1, l_2 \in \mathbb{N}_0 \setminus p(H)\} = A_2(H) \setminus \{87, 92, 99, 100\}.$$

Hence, we get

$$\sharp L_2(p(H)) = \sharp L_2(H) - 4 = 91 = 3 \times 31 - 2 = 3g(p(H)) - 2.$$

Thus, three primitive semigroups H , $d_2(H)$ and $p(H)$ are Buchwietz. Namely, the diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) & & \end{array}$$

is Buchweitz.

The following is a key lemma to get a Buchweitz diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) = H_0 & & \end{array}$$

from a given Buchweitz primitive numerical semigroup H_0 with $\sharp L_2(H_0) \geq 3g(H_0) - 2$:

Lemma 2.2. *Let H_0 be a primitive n -semigroup of genus g_0 with $g_0 \geq n + 6$. Let H be a primitive $2n$ -semigroup such that*

$$\mathbb{N}_0 \setminus H = \{1 \longrightarrow 2n - 1\} \cup \{2l_n, 2l_{n+1}, \dots, 2l_{g_0}\} \cup \{4n - 3, 4n - 1\}$$

where $\mathbb{N}_0 \setminus H_0 = \{1 \longrightarrow n - 1, l_n < \dots < l_{g_0}\}$. Then the following hold:

(i) We have $d_2(H) = H_0$.

(ii) If $\sharp L_2(H_0) \geq 3g_0 - 2$, then $\sharp L_2(H) \geq 3g(H) - 2$ and $\sharp L_2(p(H)) \geq 3g(p(H)) - 1$.

Proof. The statement (i) is trivial.

We will prove (ii). If we set $\mathbb{N}_0 \setminus H_0 = \{l_1 < \dots < l_{n-1} < l_n < \dots < l_{g_0}\}$ with $l_i = i$ for $i = 1, \dots, n - 1$, then we have

$$\mathbb{N}_0 \setminus H = \{2l_1 < \dots < 2l_{n-1} < 2l_n < \dots < 2l_{g_0}\} \cup \{1, 3, \dots, 2n - 1\} \cup \{4n - 3, 4n - 1\}$$

where $2l_{g_0} \leq 2(2n - 1)$. We note that

$$\sharp\{2l_i + 2l_j \mid i, j \in \{1, 2, \dots, g_0\}\} = \sharp\{l_i + l_j \mid i, j \in \{1, 2, \dots, g_0\}\} = \sharp L_2(H_0) \geq 3g_0 - 2$$

and $g(H) = g_0 + n + 2$. Moreover, we have

$$\mathbb{N}_0 \setminus p(H) = \{2l_1 < \dots < 2l_{n-1} < 2l_n < \dots < 2l_{g_0}\} \cup \{1, 3, \dots, 2n - 1\} \cup \{4n - 3\}$$

and $g(p(H)) = g_0 + n + 1$. Now we have

$$\sharp\{2i + (2j + 1) \mid i = 1, 2, \dots, n - 1, j = 0, 1, \dots, n - 1\} = \sharp\{2l + 1 \mid l = 1, 2, \dots, 2n - 2\} = 2n - 2.$$

We note that $2l + 1 \leq 4n - 3$ for $l \leq 2n - 2$. On the other hand, we obtain

$$\sharp\{2l_i + (4n - 3) \mid i = 1, 2, \dots, g_0\} = g_0 \geq n + 6.$$

In view of $2l_i + (4n - 3) \geq 4n - 1 > 4n - 3$ we get

$$\sharp L_2(p(H)) \geq 3g_0 - 2 + 2n - 2 + n + 6 = 3(g_0 + n + 1) - 1 = 3g(p(H)) - 1.$$

The set $L_2(H)$ contains

$$L_2(p(H)) \cup \{(4n - 1) + 2l_{g_0}, (4n - 1) + (4n - 3), (4n - 1) + (4n - 1)\}.$$

Then $8n - 2 \notin L_2(p(H))$, because of

$$2l_{g_0} + 2l_{g_0} \leq 2(2n - 1) + 2(2n - 1) = 8n - 4.$$

Moreover, we have $4n - 1 + 2l_{g_0} \notin L_2(p(H))$. Thus, we get

$$\sharp L_2(H) \geq \sharp L_2(p(H)) + 2 \geq 3g(p(H)) - 1 + 2 = 3(g_0 + n + 1) + 1 = 3(g_0 + n + 2) - 2 = 3g(H) - 2.$$

□

There are many examples of Buchweitz numerical semigroups H_0 with $\sharp L_2(H_0) \geq 3g(H_0) - 2$. For example, if we use such numerical semigroups of Proposition 2.2 (1) with $m = 6$ in 5), we get Buchweitz diagrams as follows:

Example 2.2. Let n be an integer with $n \geq 20$. Let H_0 be a primitive n -semigroup such that

$$\mathbb{N}_0 \setminus H_0 = \{1 \longrightarrow n - 1, 2n - 13, 2n - 11, 2n - 9, 2n - 7, 2n - 5, 2n - 2, 2n - 1\}.$$

Then we have $\sharp L_2(H_0) = 3g(H_0) - 2$ and $g_0 = g(H_0) = n - 1 + 7 = n + 6$. Let H be a primitive $2n$ -semigroup such that

$$\mathbb{N}_0 \setminus H = \{1 \longrightarrow 2n - 1\} \cup \{4n - 26, 4n - 22, 4n - 18, 4n - 14, 4n - 10, 4n - 4, 4n - 2\} \cup \{4n - 3, 4n - 1\}.$$

By Lemma 2 we have a Buchweitz diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) = H_0 & & \end{array}$$

3. Non-Buchweitz diagrams

In this section we give several kinds of non-Buchweitz diagrams

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) & & \end{array}$$

with a Buchweitz numerical semigroup H .

Example 3.1. Let $H = \langle 13 \rightarrow 18, 20, 22, 23 \rangle$, which is a Buchweitz numerical semigroup. Then we have $d_2(H) = \langle 7 \rightarrow 11, 13 \rangle$, which is Weierstrass⁶⁾. Hence, $d_2(H)$ is non-Buchweitz. Thus, we get a non-Buchweitz diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) & & \end{array}$$

Example 3.2. Let n be an integer with $n \geq 20$. Let H be a primitive n -semigroup such that

$$\mathbb{N}_0 \setminus H = \{1 \rightarrow n-1, 2n-13, 2n-11, 2n-9, 2n-7, 2n-5, 2n-2, 2n-1\},$$

which appears in Example 2.2. In this case, we have

$$\mathbb{N}_0 \setminus d_2(H) = \{1 \rightarrow \left\lfloor \frac{n-1}{2} \right\rfloor\} \cup \{n-1\},$$

where for a real number r the symbol $\lfloor r \rfloor$ means the largest integer which is less than or equal to r . The semigroup H is Buchweitz and $d_2(H)$ is Weierstrass⁶⁾. Thus, we get a non-Buchweitz diagram.

In the above examples, we do not mention the property of the numerical semigroup $p(H)$. In the following example $p(H)$ is Buchweitz and $d_2(H)$ is non-Buchweitz.

Example 3.3. Let H be a primitive 112-semigroup of genus 118 such that

$$\mathbb{N}_0 \setminus H = \{1 \rightarrow 111\} \cup \{121, 145, 215, 219, 221, 223\} \cup \{222\}.$$

In this case, we have $\mathbb{N}_0 \setminus d_2(H) = \{1 \rightarrow 55, 111\}$. Hence, the numerical semigroup $d_2(H)$ is Weierstrass, which implies that it is non-Buchweitz. Moreover, we have

$$\mathbb{N}_0 \setminus p(H) = \{1 \rightarrow 111\} \cup \{121, 145, 215, 219, 221\} \cup \{222\}.$$

First, we will show that $\sharp L_2(H) = 3g(H) + 1$. We have

$$A(H) = \{\gamma - 1 \mid \gamma \in \mathbb{N}_0 \setminus H\} = \{0 \rightarrow 110\} \cup \{120, 144, 214, 218, 220, 222\} \cup \{221\}.$$

We set $A_2(H) = \{a + a' \mid a, a' \in A(H)\}$. Then the set $A_2(H)$ consists of the following elements:

$$\begin{aligned} 0 &= 0 + 0 \rightarrow 0 + 110 = 110, 111 = 110 + 1 \rightarrow 110 + 110 = 220, 221 = 221 + 0 \rightarrow 221 + 110 = 331, \\ 332 &= 110 + 222, 334 = 120 + 214, 338 = 120 + 218, 340 = 120 + 220, 341 = 120 + 221, 342 = 120 + 222, \\ 358 &= 144 + 214, 362 = 144 + 218, 364 = 144 + 220, 365 = 144 + 221, 366 = 144 + 222, \\ 428 &= 214 + 214, 432 = 214 + 218, 434 = 214 + 220, 435 = 214 + 221, 436 = 214 + 222, 438 = 218 + 220, \\ 439 &= 218 + 221, 440 = 218 + 222, 441 = 220 + 221, 442 = 220 + 222, 443 = 221 + 222, 444 = 222 + 222. \end{aligned}$$

Hence, we get

$$\sharp L_2(H) = \sharp A_2(H) = 332 + 23 = 355 = 3 \times 118 + 1 = 3g(H) + 1.$$

Second, we prove that $\sharp L_2(p(H)) = 3g(p(H)) - 1$. Seeing the elements of $A_2(H)$ we obtain

$$A_2(p(H)) = A_2(H) \setminus \{332, 342, 366, 443, 444\}$$

where $A_2(p(H)) = \{(l-1) + (l'-1) \mid l, l' \in \mathbb{N}_0 \setminus p(H)\}$, because we have

$$436 = 218 + 218, 440 = 220 + 220, 442 = 221 + 221.$$

Hence, we get

$$\sharp L_2(p(H)) = \sharp A_2(p(H)) = 355 - 5 = 350 = 3 \times 117 - 1 = 3g(p(H)) - 1.$$

Thus, we obtain a non-Buchweitz diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) & & \end{array}$$

where H and $p(H)$ are Buchweitz, and $d_2(H)$ is non-Buchweitz.

Finally, we give examples of Buchweitz numerical semigroups H such that $p(H)$ is non-Buchweitz.

Example 3.4. Let n be an integer with $n \geq 97$. Consider a primitive n -semigroup H whose complement $\mathbb{N}_0 \setminus H$ is

$$\{1 \longrightarrow n-1, 2n-25, 2n-5, 2n-1\}.$$

In this case, the genus of H is $n+2$. By Example 4.6 in 5) we have

$$\sharp L_2(H) \leq 3g(H) - 3, \sharp L_3(H) \leq 5g(H) - 5 \text{ and } \sharp L_4(H) = 7g(H) - 6 > 7g(H) - 7.$$

Hence, the semigroup H is Buchweitz. On the other hand the complement $\mathbb{N}_0 \setminus p(H)$ of the semigroup $p(H)$ is

$$\{1 \longrightarrow n-1, 2n-25, 2n-5\}.$$

By Theorem 1.6 in 5) we have

$$\sharp L_m(p(H)) \leq (2m-1)(g(p(H)) - 1)$$

for all $m \geq 2$. Hence, $p(H)$ is a non-Buchweitz numerical semigroup. Thus, we get a non-Buchweitz diagram

$$\begin{array}{ccc} H & \xrightarrow{p} & p(H) \\ \downarrow d_2 & & \\ d_2(H) & & \end{array}$$

with Buchweitz H and non-Buchweitz $p(H)$. In this case, $d_2(H)$ is non-Buchweitz. In fact, the integers $2n-25$, $2n-5$ and $2n-1$ in $\mathbb{N}_0 \setminus H$ are odd. Hence, we obtain

$$d_2(H) = \{1 \longrightarrow \left\lfloor \frac{n-1}{2} \right\rfloor\}.$$

Thus, $d_2(H)$ is the Weierstrass semigroup of an ordinary point of a curve. Hence, $d_2(H)$ is non-Buchweitz.

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