

[研究論文] Numerical semigroups and affine spaces

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Abstract

In this paper we show that every symmetric numerical semigroup H generated by three elements is of affine space type where a numerical semigroup is said to be of *affine space type* if a minimal embedding of its monomial curve into the affine space is derived from an embedding of some affine space into an affine space by substituting monomials. In the case of a symmetric numerical semigroup generated by four elements we give an easy computable sufficient condition to be of affine space type.

Keywords: Numerical semigroup, Symmetric numerical semigroup, Telescopic numerical semigroup

1. Introduction

A *numerical semigroup* H is a submonoid of the additive monoid \mathbb{N}_0 of non-negative integers such that its complement $\mathbb{N}_0 \setminus H$ in \mathbb{N}_0 is finite. The cardinality of $\mathbb{N}_0 \setminus H$ is called the *genus* of H , which is denoted by $g(H)$. Let $M(H)$ be the minimal set of generators for the additive monoid H , which is uniquely determined. We set $M(H) = \{a_1, \dots, a_n\}$. Then the greatest common divisor of a_1, a_2, \dots, a_n is 1. Let k be an algebraically closed field of characteristic 0. Then we can embed the monomial curve $C_H = \text{Spec } k[H]$ associated to H into the n -dimensional affine space through ${}^a\varphi_H : C_H \hookrightarrow \mathbb{A}^n$ defined by the k -algebra homomorphism $\varphi_H : k[X_1, \dots, X_n] \longrightarrow k[H] = k[t^h]_{h \in H}$ sending X_i to t^{a_i} for each i . We denote by I_H the ideal of the kernel of φ_H . Then the k -algebra $k[H]$ is isomorphic to $k[X_1, \dots, X_n]/I_H$. The numerical semigroup H is said to be of *affine space type* if there exists an embedding ι from the affine space \mathbb{A}^{m+1-n} of dimension $m+1-n$ into the m -dimensional affine space \mathbb{A}^m such that we have the fiber product

$$\begin{array}{ccc} C_H & \xrightarrow{{}^a\varphi_H} & \mathbb{A}^n \\ \downarrow & \square & \downarrow \zeta \\ \mathbb{A}^{m+1-n} & \xrightarrow{\iota} & \mathbb{A}^m \end{array}$$

of \mathbb{A}^{m+1-n} and \mathbb{A}^n over \mathbb{A}^m through some morphism ζ from \mathbb{A}^n to \mathbb{A}^m sending (x_1, \dots, x_n) to

$$(M_1(x_1, \dots, x_n), \dots, M_m(x_1, \dots, x_n))$$

where $M_i(x_1, \dots, x_n)$'s are non-constant monomials.

We investigate numerical semigroups H with $\sharp M(H) = 2$ in Section 2. We show that H is of affine space type. We generalize these H 's to numerical semigroups with $\sharp M(H) = n \geq 3$. We set $c(H) = \min\{n \in \mathbb{N}_0 \mid n + \mathbb{N}_0 \subseteq H\}$, which is called the *conductor* of H . It is well-known that $c(H) \leq 2g(H)$. A numerical semigroup is said to be *symmetric* if $c(H) = 2g(H)$. We prove that a symmetric numerical semigroup H with $\sharp M(H) = 3$ is of affine space type in Section 3. Moreover, we generalize these H 's to numerical semigroups with $\sharp M(H) = n \geq 4$, which is said to be *telescopic*. We show that any telescopic numerical semigroup is of affine space type. In Section 4 we give a sufficient condition for a symmetric numerical semigroup generated by four elements to be telescopic. Moreover, we see that there is a symmetric numerical semigroup generated by four elements which is not telescopic.

2. Numerical semigroups generated by two elements

Let H be a numerical semigroup with $M(H) = \{a_1, a_2\}$. Then we have $I_H = (X_1^{a_2} - X_2^{a_1})$. Hence, we have the fiber product

$$\begin{array}{ccc} C_H & \xrightarrow{\varphi_H} & \mathbb{A}^2 \\ \downarrow & \square & \downarrow \zeta \\ \mathbb{A}^1 & \xrightarrow{\iota} & \mathbb{A}^2 \end{array}$$

of \mathbb{A}^1 and \mathbb{A}^2 over \mathbb{A}^2 through the embedding ι sending y to (y, y) and the morphism ζ sending (x_1, x_2) to $(x_1^{a_2}, x_2^{a_1})$. Thus, H is of affine space type. For any positive integers a_1, \dots, a_n we denote by (a_1, \dots, a_n) (reps. $[a_1, \dots, a_n]$) the greatest common divisor (reps. the least common multiple) of a_1, \dots, a_n . For any positive integers a_1, \dots, a_n we denote by $\langle a_1, \dots, a_n \rangle$ the additive monoid generated by a_1, \dots, a_n . Numerical semigroups generated by two elements can be generalized to the following numerical semigroups:

Theorem 2.1. *Let $n \geq 2$ and $2 \leq r_n < r_{n-1} < \dots < r_2 < r_1$ with $(r_i, r_j) = 1$ if $i \neq j$. We set $a_i = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n$ for $i = 1, \dots, n$. Let $H = \langle a_1, a_2, \dots, a_n \rangle$. Then we have the following:*

- (i) $(a_1, a_2, \dots, a_n) = 1$, hence H is a numerical semigroup.
- (ii) $M(H) = \{a_1, a_2, \dots, a_n\}$.
- (iii) H is symmetric.
- (iv) I_H is generated by the set $\{X_i^{r_i} - X_{i+1}^{r_{i+1}} \mid i = 1, 2, \dots, n-1\}$.

Proof. (i) Assume that $m = (a_1, \dots, a_n) > 1$. We set $m = m'p$ where p is a prime number. Then we have $p|a_i$ for all i . Hence, $p|a_1$, which implies that $p|r_j$ for some $j \neq 1$. Moreover, we get $p|a_j$, which implies that $p|r_k$ for some $k \neq j$. Hence, we have $p|(r_j, r_k)$, which contradicts $(r_j, r_k) = 1$.

(ii) Assume that $a_i \in \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$, which implies that

$$r_1 \cdots r_{i-1} r_{i+1} \cdots r_n = a_i = \beta_1 a_1 + \cdots + \beta_{i-1} a_{i-1} + \beta_{i+1} a_{i+1} + \cdots + \beta_n a_n = r_i q$$

for some integer q . Hence, we obtain $r_i | (r_1 \cdots r_{i-1} r_{i+1} \cdots r_n)$, which contradicts $(r_i, r_j) = 1$ for any $j \neq i$.

(iii) We have

$$(a_1, a_2, \dots, a_i) = (r_2 \cdots r_n, r_1 r_3 \cdots r_n, \dots, r_1 \cdots r_{i-1} r_{i+1} \cdots r_n) = r_{i+1} \cdots r_n$$

because of $(r_l, r_j) = 1$ if $l \neq j$. Hence, we get

$$\begin{aligned} [(a_1, \dots, a_i), a_{i+1}] &= [r_{i+1} \cdots r_n, r_1 \cdots r_i r_{i+2} \cdots r_n] = \frac{r_{i+1} \cdots r_n r_1 \cdots r_i r_{i+2} \cdots r_n}{(r_{i+1} \cdots r_n, r_1 \cdots r_i r_{i+2} \cdots r_n)} \\ &= \frac{r_{i+1} \cdots r_n r_1 \cdots r_i r_{i+2} \cdots r_n}{r_{i+2} \cdots r_n} = r_1 \cdots r_n = r_1 a_1 \in \langle a_1, \dots, a_i \rangle. \end{aligned}$$

By Theorem 2.1 and Definition 2.2 in 1) H is symmetric.

(iv) Let

$$m_i a_i = \beta_1 a_1 + \cdots + \beta_{i-1} a_{i-1} + \beta_{i+1} a_{i+1} + \cdots + \beta_n a_n$$

where m_i is a positive integer and β_j 's are non-negative integers. Hence, we get

$$m_i r_1 \cdots r_{i-1} r_{i+1} \cdots r_n = r_i \left(\beta_1 \frac{a_1}{r_i} + \cdots + \beta_{i-1} \frac{a_{i-1}}{r_i} + \beta_{i+1} \frac{a_{i+1}}{r_i} + \cdots + \beta_n \frac{a_n}{r_i} \right),$$

which implies that $r_i | m_i$. Hence, we obtain $m_i \geq r_i$. Thus, we have minimal relations $r_i a_i = r_j a_j$ for all $i \neq j$. By 1) the ideal I_H is generated by $X_i^{r_i} - X_{i+1}^{r_{i+1}}$ ($i = 1, \dots, n-1$). \square

Example 2.1. Let $r_1 = 5$, $r_2 = 3$ and $r_3 = 2$ in Theorem 2.1. Then we have $a_1 = 6$, $a_2 = 10$ and $a_3 = 15$. Hence the numerical semigroup $\langle 6, 10, 15 \rangle$ is symmetric. The ideal I_H is generated by $X_1^5 - X_2^3$ and $X_2^3 - X_3^2$.

Corollary 2.2. *Let $H = \langle a_1, \dots, a_n \rangle$ be the numerical semigroup as in Theorem 2.1. Then H is of affine space type.*

Proof. We have $I_H = (X_1^{r_1} - X_2^{r_2}, X_2^{r_2} - X_3^{r_3}, \dots, X_{n-1}^{r_{n-1}} - X_n^{r_n})$. Hence, we have the fiber product

$$\begin{array}{ccc} C_H & \xrightarrow{a\varphi_H} & \mathbb{A}^n \\ \downarrow & \square & \downarrow \zeta \\ \mathbb{A}^1 & \xrightarrow{\iota} & \mathbb{A}^n \end{array}$$

of \mathbb{A}^1 and \mathbb{A}^n over \mathbb{A}^n through the embedding ι sending y to (y, y, \dots, y) and the morphism ζ sending (x_1, x_2, \dots, x_n) to $(x_1^{r_1}, x_2^{r_2}, \dots, x_n^{r_n})$. Thus, H is of affine space type. \square

3. Symmetric numerical semigroups generated by three elements

Let H be a symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. Herzog 1) showed that renumbering a_1, a_2 and a_3 we have minimal relations $\alpha_1 a_1 = \alpha_2 a_2$ and $\alpha_3 a_3 = \beta_1 a_1 + \beta_2 a_2$. Hence, the ideal I_H is generated by $X_1^{\alpha_1} - X_2^{\alpha_2}$ and $X_3^{\alpha_3} - X_1^{\beta_1} X_2^{\beta_2}$. If $\beta_1 \beta_2 \neq 0$, then we have the fiber product

$$\begin{array}{ccc} C_H & \xrightarrow{a\varphi_H} & \mathbb{A}^3 \\ \downarrow & \square & \downarrow \zeta \\ \mathbb{A}^3 & \xrightarrow{\iota} & \mathbb{A}^5 \end{array}$$

of \mathbb{A}^3 and \mathbb{A}^3 over \mathbb{A}^5 through the embedding ι sending (y_1, y_2, y_3) to $(y_1, y_1, y_2 y_3, y_2, y_3)$ and the morphism ζ sending (x_1, x_2, x_3) to $(x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_1^{\beta_1}, x_2^{\beta_2})$. Thus, H is of affine space type. If $\beta_1 \beta_2 = 0$, then we may assume that $\beta_2 = 0$. Then we have the fiber product

$$\begin{array}{ccc} C_H & \xrightarrow{a\varphi_H} & \mathbb{A}^3 \\ \downarrow & \square & \downarrow \zeta \\ \mathbb{A}^2 & \xrightarrow{\iota} & \mathbb{A}^4 \end{array}$$

of \mathbb{A}^2 and \mathbb{A}^3 over \mathbb{A}^4 through the embedding ι sending (y_1, y_2) to (y_1, y_1, y_2, y_2) and the morphism ζ sending (x_1, x_2, x_3) to $(x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}, x_1^{\beta_1})$. Therefore, H is of affine space type.

Example 3.1. Let H be a numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$ where $a_1 = 4, a_2 = 6$ and $a_3 = 5$. We have $g(H) = 4$ and $c(H) = 7$. Hence, H is symmetric. Moreover, the ideal I_H is generated by $X_1^3 - X_2^2$ and $X_3^2 - X_1 X_2$.

Example 3.2. Let H be a numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$ where $a_1 = 6, a_2 = 9$ and $a_3 = 8$. We have $g(H) = 10$ and $c(H) = 19$. Hence, H is symmetric. Moreover, the ideal I_H is generated by $X_1^3 - X_2^2$ and $X_3^3 - X_1^4$.

To generalize a symmetric numerical semigroup generated by three elements we consider the following numerical semigroup: Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. For any $i = 1, 2, \dots, n$ let d_i be the greatest common divisor among a_1, \dots, a_i . The numerical semigroup H is said to be *telescopic* if renumbering a_1, \dots, a_n we have $\frac{a_{i+1}}{d_{i+1}} \in \langle \frac{a_1}{d_1}, \dots, \frac{a_i}{d_i} \rangle$ for all $i = 1, \dots, n-1$. Hence, any telescopic numerical semigroup H with $M(H) = \{a_1, \dots, a_n\}$ is symmetric. In fact, by definition we have

$$a_{i+1} \in \langle \frac{d_{i+1} a_1}{d_i}, \dots, \frac{d_{i+1} a_i}{d_i} \rangle,$$

which implies that

$$\langle a_1, \dots, a_n \rangle \ni \frac{d_i a_{i+1}}{d_{i+1}} = [d_i, a_{i+1}] = [(a_1, \dots, a_i), a_{i+1}].$$

Hence, by Theorem 2.1 and Definition 2.2 in 1) H is symmetric. The notion of a telescopic numerical semigroup is a generalization of a symmetric numerical semigroup generated by three elements (for example, see Remark 3.8 in 2)).

Theorem 3.1. Any telescopic numerical semigroup H is of affine space type.

Proof. Let $M(H) = \{a_1, \dots, a_n\}$ satisfying $\frac{a_{i+1}}{d_{i+1}} \in \langle \frac{a_1}{d_1}, \dots, \frac{a_i}{d_i} \rangle$ for all $i = 1, \dots, n-1$. We set $\alpha_{i+1} = \frac{d_i}{d_{i+1}}$ for $i = 1, \dots, n-1$. Then we can write $\alpha_{i+1} a_{i+1} = \sum_{j=1}^i \beta_{i+1,j} a_j$ with $\beta_{i+1,j} \in \mathbb{N}_0$. By Proposition 2.3 in 3) the ideal I_H

is generated by $n - 1$ polynomials $X_{i+1}^{\alpha_{i+1}} - \prod_{j=1}^i X_j^{\beta_{i+1,j}}$ with $i = 1, \dots, n - 1$. We set $l = \#\{(i + 1, j) \mid \beta_{i+1,j} \neq 0\}$.

Then we have a fiber product

$$\begin{array}{ccc} C_H & \xrightarrow{\varphi_H} & \mathbb{A}^n \\ \downarrow & \square & \downarrow \zeta \\ \mathbb{A}^l & \xrightarrow{\iota} & \mathbb{A}^{l+n-1} \end{array}$$

where we omit to describe the maps ζ and ι . Hence, H is of affine space type. \square

Proposition 3.2. *Let $H = \langle a_1, \dots, a_n \rangle$ be the numerical semigroup as in Theorem 2.1. Then H is telescopic.*

Proof. We use the notation in Theorem 2.1. Then we have

$$d_i = (r_2 \cdots r_n, r_1 r_3 \cdots r_n, \dots, r_1 \cdots r_{i-1} r_{i+1} \cdots r_n) = r_{i+1} \cdots r_n.$$

Hence, we get

$$\frac{a_{i+1}}{d_{i+1}} = \frac{r_1 \cdots r_i r_{i+2} \cdots r_n}{r_{i+2} \cdots r_n} = r_1 \cdots r_i = r_1 r_2 \cdots r_i = r_1 \frac{a_1}{d_i} \in \langle \frac{a_1}{d_i}, \dots, \frac{a_i}{d_i} \rangle$$

for all $i = 1, \dots, n - 1$, which implies that H is telescopic. \square

4. Symmetric numerical semigroups generated by four elements which are telescopic

For a numerical semigroup H we denote by $d_2(H)$ the set of the elements $\frac{h}{2}$ where h is an even number in H . Then $d_2(H)$ is a numerical semigroup. In this section we are interested in symmetric numerical semigroups H generated by four elements. Here, we describe a sufficient condition for H to be telescopic in terms of $d_2(H)$.

Theorem 4.1. *Let H be a symmetric numerical semigroup with $\sharp M(H) = 4$. If $d_2(H)$ is a symmetric numerical semigroup with $g(H) \geq 3g(d_2(H))$, then H is telescopic, hence it is of affine space type. In this case, we have $\sharp M(d_2(H)) = 3$.*

Proof. Let $d_2(H) = \mathbb{N}_0$. Then we have $H \ni 2$, which implies that $H = \langle 2, 2g + 1 \rangle$. This contradicts $\sharp M(H) = 4$.

Let $\sharp M(d_2(H)) = 2$. We set $d_2(H) = \langle a, b \rangle$. Since H is symmetric, by Proposition 2.5 in 4) we have

$$H = 2d_2(H) \cup \{2g - 1 - 2t \mid t \in \mathbb{Z} \setminus d_2(H)\}$$

where g is the genus of H . The maximum element of $\mathbb{Z} \setminus d_2(H)$ is $2g(d_2(H)) - 1 = (a - 1)(b - 1) - 1$, because $d_2(H) = \langle a, b \rangle$ is symmetric and $g(\langle a, b \rangle) = \frac{(a-1)(b-1)}{2}$. Hence, we get

$$M(H) \ni 2a, 2b, 2g - 1 - (2(a - 1)(b - 1) - 1).$$

Moreover, since $d_2(H)$ is symmetric, by Lemma 2.1 in 4) an element $t \in \mathbb{Z} \setminus d_2(H)$ is described by $t = (a - 1)(b - 1) - 1 - h$ with $h \in d_2(H)$. Thus, we obtain

$$2g - 1 - 2t = 2g - 1 - 2((a - 1)(b - 1) - 1 - h) = 2h + (2g - 1 - (2(a - 1)(b - 1) - 1)).$$

Hence, H is generated by $2a, 2b$ and $2g - 1 - (2(a - 1)(b - 1) - 1)$, which contradicts $\sharp M(H) = 4$. Thus, we must have $\sharp M(d_2(H)) = 3$.

Let $M(d_2(H)) = \{a_1, a_2, a_3\}$ with $a_i \neq a_j$ if $i \neq j$. Since $d_2(H)$ is symmetric, by Section 3 $d_2(H)$ is telescopic. Hence, we may assume that $a_3 \in \langle \frac{a_1}{d_2}, \frac{a_2}{d_2} \rangle$ where d_2 is the greatest common divisor of a_1 and a_2 . By Proposition 2.5 in 4) we have

$$M(H) = \{2a_1, 2a_2, 2a_3, 2g - 1 - 2(2g(d_2(H)) - 1)\},$$

because $d_2(H)$ is symmetric. which implies that the maximum element of $d_2(H)$ is $2g(d_2(H)) - 1$. In view of the assumption $g(H) \geq 3g(d_2(H))$ we obtain

$$2g - 1 - 2(2g(d_2(H)) - 1) \geq 6g(d_2(H)) - 1 - 4g(d_2(H)) + 2 = 2g(d_2(H)) + 1 > c(d_2(H)),$$

which implies that

$$2g - 1 - 2(2g(d_2(H)) - 1) \in d_2(H) = \langle a_1, a_2, a_3 \rangle = \langle \frac{2a_1}{d_3}, \frac{2a_2}{d_3}, \frac{2a_3}{d_3} \rangle$$

where d_3 is the greatest common divisor among $2a_1, 2a_2$ and $2a_3$, i.e., $d_3 = 2$. Thus, H is telescopic. \square

Example 4.1. Let $H = \langle 8, 12, 14, 2g - 15 \rangle$ with $g \geq 12$. Then H is a telescopic numerical semigroup. In fact, H is symmetric. Moreover, we have $d_2(H) = \langle 4, 6, 7 \rangle$, which is a symmetric numerical semigroup of genus 4. By Theorem 4.1 H is telescopic, hence it is of affine space type.

Example 4.2. Let $H = \langle 5, 6, 7, 8 \rangle$. Then we have $d_2(H) = \langle 3, 4, 5 \rangle$, which is a non-symmetric numerical semigroup of genus 2. In this case, H is a symmetric numerical semigroup of genus 5. But H is not telescopic.

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