[研究論文] Numerical semigroups which are not the Weierstrass semigroups on double covers of plane curves

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Abstract

We investigate the Weierstrass semigroups of ramification points on double covers of smooth plane curves of degree $d \geqq 5$ such that the ramification points are on flexes whose tangential multiplicities are $d-2$. Using the results we give numerical semigroups which are not the Weierstrass semigroups of the ramification points.

Keywords: Numerical semigroup, Smooth plane curve, Weierstrass semigroup, Double cover of a curve

1. Introduction

Let C be a smooth irreducible curve of genus g, where a *curve* means a projective curve over an algebraically closed field k of characteristic 0. For a point P of C we define the *Weierstrass semigroup*

$$H(P) = \{n \in \mathbb{N}_0 \mid \text{there exists a rational function } f \text{ on } C \text{ such that } (f)_\infty = nP\}$$

of P, where $\mathbb{N}_0$ is the additive monoid of non-negative integers and $(f)_\infty$ means the polar divisor of f. Then $H(P)$ is a *numerical semigroup of genus* g, which means a submonoid of $\mathbb{N}_0$ whose complement is a finite set with cardinality g.

Let C be a smooth plane curve of degree $d \geqq 4$. For a point P of C we denote by T_P the tangent line at P on C. Let Z be a plane curve. We denote by $C.Z$ the intersection divisor of C with Z. Moreover, let $\mathrm{ord}_P\, C.Z$ be the multiplicity of $C.Z$ at P. For $d \leqq 6$ we describe the semigroups $H(P)$ in 1). For a point P with $\mathrm{ord}_P\, C.T_P \geqq d-2$ the semigroup $H(P)$ is uniquely determined (see 2)). For a numerical semigroup H we denote by $d_2(H)$ the set consisting of the elements $\frac{h}{2}$ for even $h \in H$, which is a numerical semigroup. The study of this paper is related to the numerical semigroups $H = H(\tilde{P})$ which are the Weierstrass semigroups of ramification points $\tilde{P}$ on double covers π of smooth plane curves of degree d. In this case we have $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$, which is the Weierstrass semigroup of a point on a smooth plane curve of degree d.. Such a numerical semigroup H is said to be the *double covering type of a plane curve*. The paper 3) shows that every numerical semigroup H of $g(H) \geqq 9$ except one type whose $d_2(H)$ is the Weierstrass semigroup of a point on a smooth plane curve of degree 4 is the double covering type of a plane curve. The excluded semigroup is attained by a ramification point on a double cover of a hyperelliptic curve of genus 3. In 4) we showed that for any $d \geqq 5$ there exists a numerical semigroup H whose $d_2(H)$ is the Weierstrass semigroup of a point P on a smooth plane curve of degree d with $\mathrm{ord}_P\, C.T_P = d-1$ such that H is not the double covering type of a plane curve. In this paper we will prove the following:

Main Theorem. *Let $d \geqq 5$. Then for sufficiently large g there is a numerical semigroup H of genus g whose $d_2(H)$ is the Weierstrass semigroup of a point P on a smooth plane curve of degree d with* $\mathrm{ord}_P\, C.T_P = d-2$ *such that H is not the double covering type of a plane curve.*

2. Case of $\mathrm{ord}_P(C.T_P) = d-2$

To describe a numerical semigroup we use the following notation: For any positive integers $a_1, a_2, \ldots, a_l$ we denote by $\langle a_1, a_2, \ldots, a_l\rangle$ the additive monoid generated by $a_1, a_2, \ldots, a_l$. By 3) we know that any numerical semigroup H of genus $\geqq 9$ with $d_2(H) = \langle 4,5,6,7\rangle$ with odd $n \geqq 3$ is the double covering type of a plane curve except the semigroups $2\langle 4,5,6,7\rangle + \langle n, n+2\rangle$. We note that a point P on a non-hyperelliptic curve with $H(P) = \langle 4,5,6,7\rangle$ is on a smooth plane curve of degree 4 with $\mathrm{ord}_P\, C.T_P = 2$. We consider a numerical semigroup H with $d_2(H) = H(P)$, where P is a point of a smooth plane curve C of degree $d \geqq 5$ with $\mathrm{ord}_P\, C.T_P = d-2$. In this article we will investigate whether H with $g(H) = 2g(d_2(H)) + \dfrac{n-1}{2} - 1$ is the double covering type of a plane curve or not, where n is the least odd integer in H. We will show that any H except for two kinds of numerical semigroups is not the double covering type of a plane curve. To investigate a relation between the genera of H and $d_2(H)$ we introduce the following notion: For a numerical semigroup H whose minimum positive integer is m we define $S(H) = \{m, s_1, \ldots, s_{m-1}\}$, where we set $s_i = \min\{h \in H \mid h \equiv i \bmod m\}$ for all i with $1 \leqq i \leqq m-1$. The set $S(H)$ is called the *satndard basis* for H, which is a set of generators for H.

Lemma 2.1. Let C be a smooth plane curve of degree $d \geqq 5$ and P a point of C with $\mathrm{ord}_P\, C.T_P = d-2$. Then

$$S(H(P)) = \{2(d-2)\} \cup \{2(d-2) + k(d-3) \mid k = 1, \ldots, d-3\}$$

$$\cup \{2(d-2) + k(d-3) + 1 \mid k = 1, \ldots, d-2\}.$$

Set $S(H(P)) = \{2(d-2), s_1, \ldots, s_{2(d-2)-1}\}$ with $s_i \equiv i \bmod 2(d-2)$. Then $s_i + s_j \notin S(H(P))$ for all i and j.

Proof. By 2) we have the above description of the standard basis $S(H(P))$. Consider the element

$$s = 2(d-2) + k(d-3) + 2(d-2) + l(d-3) \equiv 4 \bmod d-3.$$

We note that

$$2(d-2) + k(d-3) \equiv 2 \bmod d-3 \text{ and } 2(d-2) + k(d-3) + 1 \equiv 3 \bmod d-3.$$

If $d \geqq 6$, then the element s does not belong to $S(H(P))$, because the remainders of the above three integers divided by $d-3$ are different. If $d = 5$, then

$$s = 2(d-2) + (k+l+3)(d-3)$$

with $k+l+3 \geqq 5 > d-3 = 2$, which is not in $S(H(P))$.

Consider

$$s' = 2(d-2) + k(d-3) + 2(d-2) + l(d-3) + 1 \equiv 5 \bmod d-3.$$

If $d \geqq 7$, then s' does not belong to $S(H(P))$. If $d = 6$, then

$$s' = 2(d-2) + (k+l+3)(d-3)$$

with $k+l+3 \geqq 5 > d-3 = 3$, which is not in $S(H(P))$. If $d = 5$, then

$$s' = 2(d-2) + (k+l+3)(d-3) + 1$$

with $k+l+3 \geqq 5 > d-2 = 3$, which is not in $S(H(P))$.

Lastly, consider

$$s'' = 2(d-2) + k(d-3) + 1 + 2(d-2) + l(d-3) + 1 \equiv 6 \bmod d-3.$$

If $d \geqq 8$, then s'' does not belong to $S(H(P))$. If $d = 7$, then

$$s'' = 2(d-2) + (k+l+3)(d-3)$$

with $k+l+3 \geqq 5 > d-3 = 4$, which is not in $S(H(P))$. If $d = 6$, then

$$s'' = 2(d-2) + (k+l+3)(d-3) + 1$$

with $k+l+3 \geqq 5 > d-2 = 4$, which is not in $S(H(P))$. If $d = 5$, then

$$s'' = 2(d-2) + (k+l+4)(d-3)$$

with $k+l+4 \geqq 6 > d-3 = 2$, which is not in $S(H(P))$. □

We set

$$H_d = 2(d-2)\mathbb{N}_0 + \sum_{i=1}^{d-3}(2(d-2)+i(d-3))\mathbb{N}_0 + \sum_{i=1}^{d-2}(2(d-2)+i(d-3)+1)\mathbb{N}_0.$$

Theorem 2.2. *Let $d \geqq 5$ and n an odd number which is larger than or equal to $(d-2)(d-1)+3$.*
A numerical semigroup H is one of the following:
i) $2H_d + \langle n, n+2(d-3)+2\rangle$,
ii) $2H_d + \langle n, n+2(d-2)(d-3)+2\rangle$.
Then the semigroup H is the double covering type of a plane curve.

Proof. Let (C, P) be a pointed smooth plane curve with $H(P) = H_d$. Then $T_P.C = (d-2)P + R_1 + R_2$ with $R_i \neq P$, $i = 1, 2$. By Lemma 2.2 in 6) and Lemma 2.1 we get $g(H) = 2g(H_d) + \frac{n-1}{2} - 1$ because

$$2(d-3)+2 = 2(2(d-2)+(d-3)+1) - 4(d-2)$$

$$\text{and } 2(d-2)(d-3)+2 = 2(2(d-2)+(d-2)(d-3)+1) - 4(d-2).$$

We set $D = \frac{n+1}{2}P - Q$ with $Q \neq P$. Then the assumption on n implies that $2D - P$ is very ample, which implies that the divisor $2D - P$ is linearly equivalent to some reduced divisor not containing P. Hence, we get a double covering

$$\pi : \tilde{C} = \operatorname{Spec}(\mathcal{O}_C \oplus \mathcal{O}_C(-D)) \longrightarrow C$$

with a ramification point $\tilde{P}$ over P. In this case, we get $n \in H(\tilde{P})$.
(i) Let $Q = R_1$. Then we obtain $h^0(K-(d-3)P-Q) = h^0(K-(d-2)P-Q)$. Indeed, we consider a curve C_{d-3} of degree $d-3$ with $C_{d-3}.C \geqq (d-3)P + Q$, because of the fact that $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(C_{d-3})) \simeq H^0(C, \mathcal{O}_C(K))$. Then $C_{d-3}.T_P \geqq (d-3)P + Q$. In view of $d \geqq 5$ we get $C_{d-3} = T_P C_{d-4}$, where C_{d-4} is a curve of degree $d-4$. Hence, we get $C_{d-3}.C = T_P.C + C_{d-4}.C \geqq (d-2)P + Q$. By Proposition 2.1 in 5) we have

$$h^0((n+2(d-3)+3)\tilde{P}) = h^0(\frac{n+2(d-3)+3}{2}P) + h^0((d-2)P+Q)$$

and

$$h^0((n+2(d-3)+1)\tilde{P}) = h^0(\frac{n+2(d-3)+1}{2}P) + h^0((d-3)P+Q).$$

Thus, we obtain $n + 2(d-3) + 2 \in H(\tilde{P})$.
(ii) Let $Q \neq R_i$ for $i = 1, 2$. Let C_{d-3} be a curve of degree $d-3$ with $C_{d-3}.C \geqq (d-2)(d-3)P + Q$. Then we have $C_{d-3} = T_P^{d-3}$. But we obtain

$$C_{d-3}.C = T_P^{d-3}.C = (d-3)(d-2)P + (d-3)(R_1+R_2) \ngeqq (d-2)(d-3)P + Q.$$

Thus, we get

$$0 = h^0(K-(d-2)(d-3)P-Q) = h^0(K-((d-2)(d-3)+1)P-Q).$$

Hence, we have $n + 2(d-2)(d-3) + 2 \in H(\tilde{P})$. □

Theorem 2.3. *Let $d \geqq 5$ and n an odd number $\geqq (d-2)(d-1)+3$. A numerical semigroup H is one of the following:*
(i) $2H_d + \langle n, n+2t\rangle$ *with* $1 \leqq i \leqq d-3$, *where* $t = i(d-3)$,
(ii) $2H_d + \langle n, n+2t\rangle$ *with* $2 \leqq i \leqq d-3$, *where* $t = i(d-3)+1$.

Then the semigroup H is not the double covering type of a plane curve.

Proof. By Lemma 2.2 in 6) and Lemma 2.1 we get $g(H) = 2g(H_d) + \dfrac{n-1}{2} - 1$. Assume that H is the double covering type of a plane curve, i.e., there is a double cover $\tilde{C}$ of a smooth plane curve C of degree d with a ramification point $\tilde{P}$ over a point P of C such that $H(\tilde{P}) = H$. Let R_1 and R_2 be as in the proof of Theorem 2.2. Then by the assumption on n there exists a point Q of C distinct from P such that $2D$ is linearly equivalent to a reduced divisor containing P, where D is $\dfrac{n+1}{2}P - Q$.
(i) We must have $h^0(K - tP - Q) = h^0(K - (t-1)P - Q)$. But, let C_{d-3} be a plane curve of degree $d-3$ with $C_{d-3} = T_P^{i-1}L_1^{d-3-i}L_Q$, where L_1 is a line through P distinct from T_P and L_Q is a line through Q with $L_Q \not\ni P$. Then we have

$$C_{d-3}.C \geqq (i-1)(d-2)P + (d-3-i)P + Q = (t-1)P + Q$$

and $C_{d-3}.C \ngeqq tP + Q$. This is a contradiction.
(ii) The equality $H^0(K - ((t-1)P + Q)) = H^0(K - (tP + Q))$ must hold. But let C_{d-3} be a plane curve of degree $d-3$ with $C_{d-3} = T_P^{i-1}L_1^{d-3-i}L$, where L_1 is a line through P distinct from T_P and L is a line. Then we have

$$\begin{aligned} C_{d-3}.C &= (i-1)((d-2)P + R_1 + R_2) + (d-i-3)L_1.C + L.C \\ &\geqq (i-1)(d-2)P + (i-1)(R_1 + R_2) + (d-i-3)P + L.C \\ &= (t-2)P + (i-1)(R_1 + R_2) + L.C, \end{aligned}$$

where $T_P.C = (d-2)P + R_1 + R_2$.
Case 1: $Q = R_j$ for some j. We may assume $Q = R_1$. Let L be a line through P with $L \neq T_P$. Hence we have $L \not\ni Q$. In this case, we get

$$C_{d-3}.C \geqq (t-1)P + (i-1)Q \geqq (t-1)P + Q,$$

because $i \geqq 2$. Moreover, we have $C_{d-3}.C \ngeqq tP + Q$.
Case 2: $Q \neq R_j$ for $j = 1, 2$. Let L be the line through P and Q. Then we have $L \neq T_P$. Hence, we get $C_{d-3}.C \geqq (t-1)P + Q$ and $C_{d-3}.C \ngeqq tP + Q$.
In both cases 1 and 2 we have a contradiction. □

3. Examples in the case $d = 5$

The following examples are given in 4):

Example 3.1. Let H be a numerical semigroup whose image by d_2 is $\langle 4, 7, 10, 13\rangle$, which is the Weierstrass semigroup of a point P on a smooth plane curve C of degree 5 with $\mathrm{ord}_P(C.T_P) = 4$. Assume that $g(H) \geqq 18$. If H is neither $2\langle 4, 7, 10, 13\rangle + \langle n, n+4\rangle$ nor $2\langle 4, 7, 10, 13\rangle + \langle n, n+12\rangle$, then it is the double covering type of a plane curve. Moreover, the excluded numerical semigroups $2\langle 4, 7, 10, 13\rangle + \langle n, n+4\rangle$ and $2\langle 4, 7, 10, 13\rangle + \langle n, n+12\rangle$ are not the double covering type of a plane curve.

By Theorem 3.3 we get the following examples:

Example 3.2. Le n an odd number $\geqq 15$. A numerical semigroup H is one of the following:
a) $2\langle 6, 8, 9, 10, 11, 13\rangle + \langle n, n+4\rangle$, b) $2\langle 6, 8, 9, 10, 11, 13\rangle + \langle n, n+8\rangle$, c) $2\langle 6, 8, 9, 10, 11, 13\rangle + \langle n, n+10\rangle$.
Then the semigroup H is not the double covering type of a plane curve. Here we note that $d_2(H) = \langle 6, 8, 9, 10, 11, 13\rangle$ is the Weierstrass semigroup of a point P on a smooth plane curve C of degree 5 with $\mathrm{ord}_P(C.T_P) = 3$.

There are other examples of numerical semigroups H with $d_2(H) = \langle 6, 8, 9, 10, 11, 13\rangle$ which are not the double covering type of a plane curve.

Example 3.3. Le n an odd number $\geqq 17$. A numerical semigroup H is one of the following:
a) $2\langle 6, 8, 9, 10, 11, 13\rangle + \langle n, n+2\rangle$, b) $2\langle 6, 8, 9, 10, 11, 13\rangle + \langle n, n+4, n+8\rangle$.
Then the semigroup H is not the double covering type of a plane curve.

Proof. We have

$$S(2\langle 6,8,9,10,11,13\rangle+\langle n\rangle)=\{12,16,18,20,22,26\}\cup\{n,n+16,n+18,n+20,n+22,n+26\}.$$

Hence, we get

$$(2\langle 6,8,9,10,11,13\rangle+\langle n,n+2\rangle)\backslash(2\langle 6,8,9,10,11,13\rangle+\langle n\rangle)=\{n+2,n+14\}$$

and

$$(2\langle 6,8,9,10,11,13\rangle+\langle n,n+4,n+8\rangle)\backslash(2\langle 6,8,9,10,11,13\rangle+\langle n\rangle)=\{n+4,n+8\}.$$

Thus, we obtain

$$g(H)=2g(d_2(H))+\frac{n-1}{2}-2=2g(\langle 6,8,9,10,11,13\rangle)+\frac{n-1}{2}-2=12+\frac{n-1}{2}-2,$$

because $g(2\langle 6,8,9,10,11,13\rangle+\langle n\rangle)=12+\dfrac{n-1}{2}$ by Lemma 2.1 in 7) and Remark 2.1 in 8). Assume that H is the double covering type of a plane curve, i.e., there is a double cover $\tilde{C}$ of a smooth plane curve C of degree 5 with a ramification point $\tilde{P}$ over a point P of C such that $H(\tilde{P})=H$. Let $\pi:\tilde{C}\longrightarrow C$ be the double covering. If we set $P=\pi(\tilde{P})$, then $\mathrm{ord}_P(C.T_P)=3$, which implies that $C.T_P=3P+R_1+R_2$, where R_1 and R_2 are distinct from P. Then by the assumption on n there exist two points Q_1 and Q_2 of C distinct from P such that $2D$ is linearly equivalent to a reduced divisor containing P, where D is $\dfrac{n+1}{2}P-Q_1-Q_2$.
a) Since $n+2\in H(\tilde{P})$, we should have

$$h^0(P+Q_1+Q_2)=h^0(Q_1+Q_2)+1.$$

However, we have $h^0(P+Q_1+Q_2)=h^0(Q_1+Q_2)=1$, because C is a smooth plane curve of degree 5. This is a contradiction.
b) In view of $n+4\in H(\tilde{P})$ we have

$$h^0(2P+Q_1+Q_2)=h^0(P+Q_1+Q_2)+1,$$

which implies that $h^0(2P+Q_1+Q_2)=2$. By Namba's Theorem (see 9) and 10)), there exists a line L with $C.L\geqq 2P+Q_1+Q_2$, which is T_P. Hence, we get $\{Q_1,Q_2\}=\{R_1,R_2\}$. But

$$h^0(K-3P-R_1-R_2)=3=h^0(K-4P-R_1-R_2)+1,$$

which implies that

$$h^0(3P+Q_1+Q_2)=h^0(4P+Q_+Q_2).$$

This contradicts $n+8\in H(\tilde{P})$. □

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