[研究論文] # Weierstrass semigroups of ramification points on double covers of curves over ordinary points

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**Abstract**

The author studies Weierstrass semigroups of ramification points on double covers of curves over ordinary points. It is showed that hyperelliptic curves and non-hyperelliptic curves are characterized by the Weierstrass semigroups.

Keywords: Numerical semigroup, Weierstrass semigroup, Double cover of a curve, Hyperelliptic curve, Non-hyperelliptic curve

## 1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid $H$ of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is a finite set. The cardinality of $\mathbb{N}_0 \backslash H$ is said to be the *genus* of $H$, which is denoted by $g(H)$. Let $C$ be a *curve*, which means a complete non-singular irreducible algebraic curve over an algebraically closed field of characteritic 0 in this article. For a pointed curve $(C, P)$ of genus $g$ we set

$$H(P) = \{h \in \mathbb{N}_0 \mid \text{ there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = hP\}.$$

Then $H(P)$ becomes a numerical semigroup of genus $g$. We call $H(P)$ the *Weierstrass semigroup* of $P$. For any positive integers $a_1, \dots, a_s$ the additive monoid generated by $a_1, \dots, a_s$ is denoted by $\langle a_1, \dots, a_s \rangle$. The following is the main result of this article:

**Theorem 1.** *Let $g$ be an integer $\geqq 3$, $r$ a positive integer $\leqq g-2$ and $n$ an odd integer with $n \geqq 2g+1+2r$. Let*

$$H = 2\langle g+1 \rightarrow 2g+1 \rangle + \langle n, n+2\cdot 1, n+2\cdot 2, \dots, n+2\cdot r \rangle.$$

*Then the following are equivalent:*
i) *$(C, P)$ is a pointed hyperelliptic curve of genus $g$ with an ordinary point $P$.*
ii) *There exists a double cover $\pi : \tilde{C} \longrightarrow C$ of a curve with a ramification point $\tilde{P}$ over $P$ with $H(\tilde{P}) = H$.*

The following theorem shows that there exists a double cover $\pi : \tilde{C} \longrightarrow C$ of a hyperelliptic curve with a ramification point $\tilde{P}$ over an ordinary point $P$ such that $H(\tilde{P})$ is distinct from the numerical semigroups in Theorem 1.

**Theorem 2.** *Let $\pi : \tilde{C} \longrightarrow C$ be a double cover of a hyperelliptic curve of genus $g$ with a ramification point $\tilde{P}$ over an ordinary point $P$. Assume that*

$$n = \min\{h \in H(\tilde{P}) \mid h \textit{ is odd}\} \geqq 2g+1 \textit{ and } g(H(\tilde{P})) = 2g + \frac{n-1}{2} - 1.$$

*Then $H(\tilde{P})$ is either*

$$2\langle g+1 \longrightarrow 2g+1 \rangle + \langle n, n+2 \rangle \textit{ or } 2\langle g+1 \longrightarrow 2g+1 \rangle + \langle n, n+2g \rangle.$$

*Conversely, if $n \geqq 2g+3$, each of the numerical semigroups is attained by a ramification point of a double cover of a hyperelliptic curve.*

Non-hyperelliptic curves are characterized by some Weierstrass semigroups on double covers as the following:

**Theorem 3.** *Let $g$ be an integer $\geqq 3$ and $n$ an odd integer with $n \geqq 2g+3$. Let $(C, P)$ be a pointed curve of genus $g$ with an ordinary point $P$. Then the following are equivalent:*
i) *The curve $C$ is non-hyperelliptic.*
ii) *There exists a double cover $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over $P$ with*

$$H(\tilde{P}) = 2\langle g+1 \rightarrow 2g+1 \rangle + \langle n, n+2\cdot l \rangle$$

*for some integer $l$ with $2 \leqq l \leqq g-1$.*

## 2 Proof of Theorem 1

First, we will prove that i) implies ii). Let us consider a divisor $D = \frac{n+1}{2}P - rP'$ on the curve $C$ with $g_2^1 \sim P + P'$ where $g_2^1$ is a unique linear system of degree 2 with projective dimension 1. Then we have

$$\deg(2D - P) = n + 1 - 2r - 1 = n - 2r \geqq 2g + 1,$$

which implies that the divisor $2D - P$ is very ample. Hence we get a double cover $\pi$ : Spec $(\mathcal{O} \oplus \mathcal{O}_C(-D)) \longrightarrow$ C with a ramification point $\tilde{P}$ over $P$. We will calculate the Weierstrass semigroup $H(\tilde{P})$ of $\tilde{P}$ using Proposition 2.1 in 1). We have

$$h^0((n-1)\tilde{P}) = h^0(\frac{n-1}{2}P) + h^0(\frac{n-1}{2}P - \frac{n+1}{2}P + rP') = h^0(\frac{n-1}{2}P) + h^0(rP' - P) = h^0(\frac{n-1}{2}P),$$

because $r \leqq g$ and $P$ is an ordinary point with $g_2^1 \sim P + P'$. Moreover, we obtain

$$h^0((n+1)\tilde{P}) = h^0(\frac{n+1}{2}P) + h^0(rP') = h^0(\frac{n+1}{2}P) + 1.$$

Thus, we get $n \in H(\tilde{P})$. Since we have

$$h^0(rP + rP') = h^0(rg_2^1) = r + 1,$$

we get $h^0(iP + rP') = i + 1$ for any $i$ with $0 \leqq i \leqq r$. Since we obtain

$$h^0((n+2i+1)\tilde{P}) = h^0((i + \frac{n+1}{2})P) + h^0(iP + rP') = h^0((i + \frac{n+1}{2})P) + i + 1,$$

we get $n + 2i \in H(\tilde{P})$ for $1 \leqq i \leqq r$. Since the genera of $\tilde{C}$ and $H$ coincide, we get $H(\tilde{P}) = H$.

From here, we will show that ii) implies i). For a numerical semigroup $H'$ we set $m = \min\{h \in H' \mid h > 0\}$ and $S(H') = \{m, s_1, \ldots, s_{m-1}\}$ where $s_i = \min\{h \in H' \mid s \equiv i \bmod m\}$ for all $i = 1, \ldots, m-1$. We note that $2r \leqq 2(g-2) < 2g+1$. We have

$$S(2\langle g+1 \to 2g+1\rangle + \langle n, n+2\cdot 1, n+2\cdot 2, \ldots, n+2\cdot r\rangle)$$
$$= \{2(g+1), 2(g+2), \ldots, 2(2g+1)\} \cup \{n, n+2\cdot 1, \ldots, n+2\cdot r, n+2(g+1+r+1) \ldots, n+2(2g+1)\}.$$

Hence, we get $g(H) = 2g + \frac{n-1}{2} - r$. Let $\tilde{C}$ = Spec $(\mathcal{O}_C \oplus \mathcal{O}_C(-D))$ where $D$ is a divisor on $C$ such that $2D$ is linearly equivalent to the branch divisor of $\pi$. Then by Riemann-Hurwitz formula we have

$$2g(H) - 2 = 2(2g-2) + \deg 2D,$$

which implies that

$$\deg D = g(H) - 1 - 2g + 2 = \frac{n+1}{2} - r.$$

Since

$$h^0((n-1)\tilde{P}) = h^0(\frac{n-1}{2}P) + h^0(\frac{n-1}{2}P - D) \text{ and } h^0((n+1)\tilde{P}) = h^0(\frac{n+1}{2}P) + h^0(\frac{n+1}{2}P - D),$$

we must have

$$h^0(\frac{n-1}{2}P - D) = 0 \text{ and } h^0(\frac{n+1}{2}P - D) = 1.$$

Hence, we obtain

$$\frac{n+1}{2}P - D \sim Q_1 + \cdots + Q_r \text{ with } Q_i \neq P \text{ for all } i.$$

Since $H = H(\tilde{P})$ contais $n + 2\cdot 1, n + 2\cdot 2, \ldots, n + 2\cdot r$ and we have

$$h^0((n + (2r+1))\tilde{P}) = h^0((r + \frac{n+1}{2})P) + h^0(rP + Q_1 + \cdots + Q_r),$$

we get

$$h^0(rP + Q_1 + \cdots + Q_r) = r + 1.$$

Thus, there exists a $g_{2r}^r$ on $C$. Since $r \leqq g - 2 < g - 1$, It follows from Clifford's theorem that $C$ is hyperelliptic. □

## 3 Proof of Theorem 2

We use the notations in the proof of Theorem 1. We note that $S(H(P)) = \{g+1, g+2, \ldots, g+i+1, \ldots, 2g+1\}$. Here we have $m = m(H) = g+1$ and $s_i = g+i+1$. Hence, we get $s_i - m = g+i+1-(g+1) = i$. Since $n \geqq 2g+1 = c(H(P)) + m - 1$ and $g(H(\tilde{P})) = 2g(H(P)) + \frac{n-1}{2} - 1$. it follows from 2) that

$$H(\tilde{P}) = 2\langle g+1 \longrightarrow 2g+1\rangle + \langle n, n+2(s_i - m)\rangle = 2\langle g+1 \longrightarrow 2g+1\rangle + \langle n, n+2i\rangle.$$

Moreover, we can describe $\tilde{C}$ = Spec $(\mathcal{O}_C \oplus \mathcal{O}_C(-D))$ where $D = \frac{n+1}{2}P - Q$ with for some $Q \in C$ which is distinct from $P$. We have

$$h^0((n+2i-1)\tilde{P}) = h^0(\frac{n+2i-1}{2}P) + h^0(\frac{n+2i-1}{2}P - \frac{n+1}{2}P + Q) = h^0(\frac{n+2i-1}{2}P) + h^0((i-1)P + Q)$$

and

$$h^0((n+2i+1)\tilde{P}) = h^0(\frac{n+2i+1}{2}P) + h^0(\frac{n+2i+1}{2}P - \frac{n+1}{2}P + Q) = h^0(\frac{n+2i+1}{2}P) + h^0(iP+Q).$$

Since $n+2i \in H(\tilde{P})$ and $n+2j \notin H(\tilde{P})$ for $1 \leqq j \leqq i-1$, we obtain $h^0((i-1)P+Q) = 1$ and $h^0(iP+Q) = 2$. Hence, we have $g^1_{i+1} \sim iP+Q$ with $i+1 \leqq g+1$.

If $i+1 \leqq g$, then by page 28 in 3) we get

$$iP+Q \sim g^1_2 + P_1 + \cdots + P_{i-1} \sim P + P' + P_1 + \cdots + P_{i-1}$$

where $g^1_2 \sim P+P'$ with $P' \neq P$. Thus, we get

$$(i-1)P + Q \sim P' + P_1 + \cdots + P_{i-1}.$$

Since $h^0((i-1)P+Q) = 1$, we get $Q = P'$ and $P_1 = \cdots = P_{i-1} = P$. For $i \geqq 2$ we have

$$1 = h^0((i-1)P+Q) = h^0(P+P'+(i-2)P) = h^0(g^1_2 + (i-2)P) = 2,$$

which is a contradiction. Hence, we get $i=1$, which implies that

$$H(\tilde{P}) = 2\langle g+1 \longrightarrow 2g+1\rangle + \langle n, n+2\rangle.$$

If $i = g$, then we obtain $H(\tilde{P}) = 2\langle g+1 \longrightarrow 2g+1\rangle + \langle n, n+2g\rangle$.

The former semigroup is attained by a hyperelliptic curve from Theorem 1. The latter semigroup is attained by both a hyperelliptic curve and a non-hyperelliptic curve. We have $K_C \sim (g-1)P + E_{g-1}$ with $\deg E_{g-1} = g-1$, $h^0(E_{g-1}) = 1$ and $E_{g-1} \ngeqq P$ where $K_C$ is a canonical divisor on $C$. We take a point $Q$ satisfying $Q \nleqq E_{g-1}$. Let $H(\tilde{P}) = 2\langle g+1 \longrightarrow 2g+1\rangle + \langle n, n+2g\rangle$. We have

$$h^0((n+2g-1)\tilde{P}) = h^0(\frac{n+2g-1}{2}P) + h^0(\frac{n+2g-1}{2}P - \frac{n+1}{2}P + Q) = h^0(\frac{n+2g-1}{2}P) + h^0((g-1)P+Q)$$

and

$$h^0((n+2g+1)\tilde{P}) = h^0(\frac{n+2g+1}{2}P) + h^0(\frac{n+2g+1}{2}P - \frac{n+1}{2}P + Q) = h^0(\frac{n+2g+1}{2}P) + h^0(gP+Q).$$

Using the property of a canonical divisor we get

$$h^0((g-1)P+Q) = g+1-g+h^0(K_C - (g-1)P - Q) = 1 + h^0(E_{g-1} - Q) = 1$$

and

$$h^0(gP+Q) = g+1+1-g+h^0(K_C - gP - Q) = 2 + h^0(E_{g-1} - P - Q) = 2.$$

Thus, we obtain $n+2g \in H(\tilde{P})$, which implies that $H(\tilde{P}) = 2\langle g+1 \longrightarrow 2g+1\rangle + \langle n, n+2g\rangle$. □

## 4 Proof of Theorem 3

First we will show that ii) implies i). Assume that ii) holds. Then the genus of $\tilde{C}$ is $2g(H) + \frac{n-1}{2} - 1$. If $C$ were hyperelliptic, this contradicts Theorem 2.

Assume that $C$ is non-hyperelliptic. Since $P$ is an ordinary point, a canonical divisor $K_C$ on $C$ is linearly equivalent to $(g-1)P + E_{g-1}$ where $E_{g-1}$ is an effective divisor of degree $g-1$ with $h^0(E_{g-1}) = 1$ and $P \nleqq E_{g-1}$. We set $D = \frac{n+1}{2}P - Q$ where $Q$ is a point of $C$ with $Q \leqq E_{g-1}$. Since $n \geqq 2g+3$, the degree of the divisor $2D - P$ is larger than or equal to $2g+1$. Hence, $2D-P$ is very ample. Therefore, we get a double cover $\pi : \tilde{C} = \mathrm{Spec}\ (\mathcal{O}_C \oplus \mathcal{O}_C(-D)) \longrightarrow C$ with a ramification point $\tilde{P}$ over $P$. We have

$$h^0((n+3)\tilde{P}) = h^0(\frac{n+3}{2}P) + h^0(P+Q).$$

Since $C$ is non-hyperelliptic, we have $h^0(P+Q) = 1$. Moreover, we have

$$h^0((n+2g-1)\tilde{P}) = h^0(\frac{n+2g-1}{2}P) + h^0((g-1)P+Q).$$

Since $Q \leqq E_{g-1}$, we obtain

$$h^0((g-1)P+Q) = g+1-g+h^0(K_C - (g-1)P - Q) = 1 + h^0(E_{g-1} - Q) = 2.$$

Hence, there exists an integer $l$ with $2 \leqq l \leqq g-1$ such that

$$h^0((l-1)P+Q) = 1 \text{ and } h^0(lP+Q) = 2.$$

This means that $n+2l \in H(\tilde{P})$. Since $g(H(\tilde{P})) = 2g + \frac{n-1}{2} - 1$, we get

$$H(\tilde{P}) = 2\langle g+1 \to 2g+1\rangle + \langle n, n+2\cdot l\rangle.$$ □

Remark. i) Let $C$ be a curve of genus 3 and $n$ an odd integer $\geqq 9$. Then $C$ is non-hyperelliptic if and only if there exists a double cover $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ satisfying $H(\tilde{P}) = 2\langle 4,5,6,7\rangle + 2\langle n, n+4\rangle$.
ii) Let $C$ be a curve of genus 4 and $n$ an odd integer $\geqq 11$. Then $C$ is non-hyperelliptic if and only if there exists a double cover $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $H(\tilde{P})$ is either $2\langle 5,6,7,8,9\rangle + 2\langle n, n+4\rangle$ or $2\langle 5,6,7,8,9\rangle + 2\langle n, n+6\rangle$. By Proposition 8.1 in 4) each semigroup of them is attained by a double cover of some non-hyperelliptic curve of genus 4.

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## References

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