

[研究論文]

# Pseudo-Frobenius numbers of numerical semigroups with high conductor

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## Abstract

The author investigates the set  $PF(H)$  of pseudo-Frobenius numbers of a numerical semigroup  $H$ . Moreover, an enlargement of the set  $PF(H)$  is studied. These sets are described when  $c(H)$  is larger than or equal to  $2g(H) - 3$ , where  $c(H)$  is the conductor of  $H$  and  $g(H)$  is the genus of  $H$ .

Keywords: Numerical semigroup, Conductor, Pseudo-Frobenius number, Almost symmetric, numerical semigroup

## 0 Introduction

Let  $\mathbb{N}_0$  be the additive monoid of non-negative integers. A submonoid  $H$  of  $\mathbb{N}_0$  is called a *numerical semigroup* if the complement  $\mathbb{N}_0 \setminus H$  is a finite set. The cardinality of  $\mathbb{N}_0 \setminus H$  is said to be the *genus* of  $H$ , which is denoted by  $g(H)$ . We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteq H\},$$

which is called the *conductor* of  $H$ . It is known that  $c(H) \leq 2g(H)$  (for example, see Lemma 2.1 (3) in [1]). A numerical semigroup  $H$  is said to be *symmetric* (resp. *quasi-symmetric*) if  $c(H) = 2g(H)$  (resp.  $c(H) = 2g(H) - 1$ ). We call  $c(H) - 1$  the *Frobenius number* of  $H$ , which is denoted by  $F(H)$ . We consider the following number. A non-negative integer  $f$  is called a *pseudo-Frobenius number* of  $H$  if  $f \notin H$ , but  $f + h \in H$  for all  $h \in H$  with  $h > 0$ . Then the Frobenius number  $F(H)$  is a pseudo-Frobenius number of  $H$ . We set

$$PF(H) = \{f \in \mathbb{N}_0 \mid f \text{ is a pseudo-Frobenius number of } H\} \text{ and } PF^*(H) = PF(H) \setminus \{c(H) - 1\}$$

For the study of numerical semigroups which are neither symmetric nor quasi-symmetric the pseudo-Frobenius numbers of  $H$  play important roles. In this article we are interested in the set  $PF^*(H)$  and its enlargement. In the cases where  $c(H) = 2g(H) - 2$  and  $2g(H) - 3$  we investigate these sets. The set  $PF(H)$  and another enlargement of  $PF(H)$  are studied in [2].

## 1 An enlargement of the set of pseudo-Frobenius numbers

A numerical semigroup  $H$  is said to be an *m-semigroup* if  $m$  is the minimum positive integer in  $H$ . In this case  $m$  is called the *multiplicity* of  $H$ , which is denoted by  $m(H)$ . For an *m-semigroup*  $H$  we set

$$s_i = \min\{h \in H \mid h \equiv i \pmod{m}\}$$

for  $m = 1, \dots, m - 1$  and

$$s_{max} = \max\{s_i \mid i = 1, \dots, m - 1\}.$$

Then we get

$$F(H) = c(H) - 1 = s_{max} - m.$$

The set  $\{m, s_1, \dots, s_{m-1}\}$  is denoted by  $S(H)$ , which is called the *standard basis* for  $H$ . To determine the set  $PF(H)$  the following is useful.

**Lemma 1.1.** *Let  $H$  be a numerical semigroup. We set  $m = m(H)$ . Then we have the following:*

i) *We obtain*

$$PF(H) = \{s_i - m \mid s_i + s_j \notin S(H) \text{ for all } j\}.$$

ii) *The cardinality of the set*

$$\overline{PF^*(H)} = \{\gamma \in \mathbb{N}_0 \setminus H \mid c(H) - 1 - \gamma \in \mathbb{N}_0 \setminus H\}$$

*is equal to  $2g(H) - c(H)$ .*

iii) *The set  $PF^*(H)$  is contained in  $\overline{PF^*(H)}$ , hence the set  $\overline{PF^*(H)}$  is an enlargement of  $PF^*(H)$ .*

**Proof.** i) First, we show that  $PF(H) \subseteq \{s_i - m \mid i = 1, \dots, m-1\}$ . Let  $\gamma \in \mathbb{N}_0 \setminus H$ . Then we have  $\gamma = s_i - \nu m$  for some  $i$ , where  $\nu$  is a positive integer. Since  $\gamma \in PF(H)$ , we get

$$\gamma + m = s_i - \nu m + m = s_i - (\nu - 1)m \in H,$$

which implies that  $\nu = 1$ . Second, we let  $s_i + s_j = s_l$  for some  $j$  and  $l$ . Then we obtain  $(s_i - m) + s_j = s_l - m \notin H$ , which implies that  $s_i - m \notin PF(H)$ . Hence, we get

$$PF(H) \subseteq \{s_i - m \mid s_i + s_j \notin S(H) \text{ for all } j\}.$$

To prove that  $s_i - m \in PF(H)$  with  $s_i + s_j \notin S(H)$  for all  $j$  it suffices to show that  $(s_i - m) + s_j \in H$ . Indeed, we have  $s_i + s_j = s_l + \nu m$  for some  $l$ , where  $\nu$  is a positive integer. Hence, we get  $(s_i - m) + s_j = s_l + (\nu - 1)m \in H$ .

ii) The set  $\mathbb{N}_0 \setminus H$  decomposes into

$$\{c(H) - 1 - h \mid h \in H, 0 \leq h \leq c(H) - 2\} \cup \{l \in \mathbb{N}_0 \setminus H \mid l = c(H) - 1 - \gamma, \gamma \in \mathbb{N}_0 \setminus H\}.$$

Thus, we get

$$\sharp \overline{PF^*(H)} = g(H) - \sharp \{c(H) - 1 - h \mid h \in H, 0 \leq h \leq c(H) - 2\} = g(H) - (c(H) - 1 - (g(H) - 1)) = 2g(H) - c(H).$$

iii) Let  $s_i - m \in PF^*(H)$ . Then we get  $s_{max} - s_i \notin H$ . Indeed, assume that  $s_{max} - s_i = h \in H$ . Then we have  $s_{max} = s_j + h$ , which implies that  $h = s_j$  for some  $j$ , because it follows that  $s_{max} - s_i \neq 0$  from

$$s_i - m \neq c(H) - 1 = s_{max} - m.$$

Hence, we obtain

$$c(H) - 1 - (s_i - m) = (s_{max} - m + 1) - 1 - (s_i - m) = s_{max} - s_i \in \mathbb{N}_0 \setminus H.$$

Thus, we have  $PF^*(H) \subseteq \overline{PF^*(H)}$ . □

The following results are given in Proposition 2.2 of 3). We give its proof as an application of Lemma 1.1

**Proposition 1.2.** *Let  $H$  be a numerical semigroup.*

i) *If  $H$  is symmetric, then we have  $PF(H) = \{c(H) - 1\} = \{2g(H) - 1\}$ .*

ii) *If  $H$  is quasi-symmetric, then we obtain*

$$PF(H) = \{c(H) - 1, g(H) - 1\} = \{2g(H) - 2, g(H) - 1\}.$$

**Proof.** i) By Lemma 1.1 ii) the set  $\overline{PF^*(H)}$  is empty, because  $c(H) = 2g(H)$ . Hence it follows from Lemma 1.1 iii) that  $PF^*(H) = \emptyset$ . Thus, we get  $PF(H) = \{c(H) - 1\}$ .

ii) We have  $c(H) - 1 - (g(H) - 1) = g(H) - 1$ , because  $c(H) = 2g(H) - 1$ . Hence, we get  $g(H) - 1 \in \mathbb{N}_0 \setminus H$ . It follows from Lemma 1 ii) that  $\overline{PF^*(H)} = \{g(H) - 1\}$ . We set  $g = g(H)$  and  $m = m(H)$ . Since  $g - 1 \in \mathbb{N}_0 \setminus H$ , by Lemma 3.1 in 4) we get  $2g - 2 - (g - 1 - m) = g - 1 + m \in H$ , which implies that  $g - 1 + m = s \in S(H)$ . Assume

that  $s + s' = s'' \in S(H)$  for some  $s' \in S(H)$ . Since  $s'' - m = s + s' - m = g - 1 + s' \neq g - 1$ , by Lemma 3.1 in 4) we get

$$2g - 2 - (s'' - m) = 2g - 2 - (g - 1 + s') = g - 1 - s' \in H,$$

which implies that  $g - 1 \in H$ . This is a contradiction. Hence, we get  $g - 1 \in PF(H)$  by Lemma 1.1 i). Thus, we get our desired result.  $\square$

For a numerical semigroup  $H$  let  $t(H)$  be the cardinality of the set  $PF(H)$ , which called the *type* of  $H$ . It is known that

$$c(H) + t(H) \leq 2g(H) + 1$$

(for example see Proposition 2.2 in 2)).  $H$  is said to be *almost symmetric* if the equality  $c(H) + t(H) = 2g(H) + 1$  holds.

**Proposition 1.3.** i) Every symmetric numerical semigroup is almost symmetric. Moreover, a numerical semigroup  $H$  is symmetric if and only if  $t(H) = 1$ .

ii) If  $H$  is a quasi-symmetric numerical semigroup, then we have  $t(H) = 2$ . Hence,  $H$  is almost symmetric.

**Proof.** i) Assume that a numerical semigroup  $H$  is symmetric. By Proposition 1.2 i) we have  $t(H) = 1$ . Hence, we get  $c(H) + t(H) = 2g(H) + 1$ , which implies that  $H$  is almost symmetric. Assume that  $t(H) = 1$ , i.e.,  $PF(H) = \{c(H) - 1\} = \{s_{\max} - m(H)\}$ . By Lemma 3.4 in 5) and Lemma 1.1 i)  $H$  is symmetric.

ii) By Proposition 1.2 ii) we have  $t(H) = 2$ . Hence we get  $c(H) + t(H) = 2g(H) + 1$ .  $\square$

The following examples show that the converse of Proposition 1.3 ii) dose not hold:

*Example 1.1.* Let  $m$  and  $\nu$  be integers with  $m \geq 3, \nu \geq 1$  and  $(m, \nu) \neq (3, 1)$ . We set

$$H_{m,\nu} = \langle m, \nu m + 1, (m - 2)\nu m + m - 1 \rangle,$$

where for positive integers  $a_1, \dots, a_n$  we denote by  $\langle a_1, \dots, a_n \rangle$  the additive monoid generated by  $a_1, \dots, a_n$ . Then we have  $t(H_{m,\nu}) = 2$ , but  $H_{m,\nu}$  is not quasi-symmetric (see Example 3.4 in 4)).

**Proposition 1.4.** Let  $H$  be a numerical semigroup.

i)  $H$  is almost symmetric if and only if  $PF^*(H) = \overline{PF^*(H)}$ .

ii) If  $\gamma \in \overline{PF^*(H)}$ , then  $c(H) - 1 - \gamma \in \overline{PF^*(H)}$ . If  $H$  is almost symmetric, then this correspondence induces a bijection of  $PF^*(H)$ .

**Proof.** i)  $H$  is almost symmetric if and only if  $\sharp PF^*(H) = t(H) - 1 = 2g(H) - c(H)$ . By Proposition 1.1 ii) and iii) we have  $PF^*(H) = \overline{PF^*(H)}$ .

ii) By the definition of  $\overline{PF^*(H)}$  and i) we reach the conclusion.  $\square$

We can generalize Lemmas 2.1 and 3.1 in 4) to the case where  $H$  is an almost symmetric numerical semigroup.

**Proposition 1.5.** Let  $H$  be a numerical semigroup. Then the following are equivalent:

i)  $H$  is almost symmetric.

ii) Let  $\gamma \notin PF^*(H)$ . Then  $\gamma \in \mathbb{Z} \setminus H$  if and only if  $c(H) - 1 - \gamma \in H$ .

**Proof.** First, we show that i) implies ii). If  $\gamma$  is negative, then we have  $\gamma \in \mathbb{Z} \setminus H$  and  $c(H) - 1 - \gamma \geq c(H)$ , which implies that  $c(H) - 1 - \gamma \in H$ . Let  $\gamma \geq 0$ . If  $c(H) - 1 - \gamma \in H$ , then  $\gamma \in \mathbb{Z} \setminus H$ , because  $c(H) - 1 \notin H$ . By Proposition 1.4 i) we have  $PF^*(H) = \overline{PF^*(H)}$ . If  $\gamma \notin PF^*(H)$  and  $\gamma \in \mathbb{N}_0 \setminus H$ , then  $c(H) - 1 - \gamma \in H$ .

Next, we show that ii) implies i). Let  $\gamma \notin PF^*(H)$ . Assume that  $\gamma \in \overline{PF^*(H)}$ . Then we have  $\gamma \in \mathbb{N}_0 \setminus H$  and  $c(H) - 1 - \gamma \in \mathbb{N}_0 \setminus H$ . This contradicts the assumption. Hence, we get  $\gamma \notin \overline{PF^*(H)}$ . Thus, we have  $PF^*(H) = \overline{PF^*(H)}$ , which implies that  $H$  is almost symmetric by Proposition 1.4 i).  $\square$

We set  $S^*(H) = S(H) \setminus \{m\}$ . In the remaining part of this section we give the relation between the elements of  $S^*(H)$  and  $\overline{PF^*(H)}$ .

**Theorem 1.6.** Let  $H$  be an  $m$ -semigroup and  $s \in S^*(H)$ . Then the following are equivalent:

i) We have  $s - m \notin \overline{PF^*(H)}$ .

ii) *There exists  $s' \in S^*(H)$  such that  $s + s' = s_{\max}$ .  
In this case, we also have  $s' - m \notin \overline{PF^*(H)}$ .*

**Proof.** We have

$$\begin{aligned}\mathbb{N}_0 \setminus H &= \{c(H) - 1 - h \mid h \in H, 0 \leq h \leq c(H) - 2\} \cup \{c(H) - 1 - \gamma \in \mathbb{N}_0 \setminus H \mid \gamma \in \mathbb{N}_0 \setminus H\} \\ &= \{c(H) - 1 - h \mid h \in H, 0 \leq h \leq c(H) - 2\} \cup \overline{PF^*(H)}.\end{aligned}$$

Assume that  $s - m \notin \overline{PF^*(H)}$ . Then we get  $s - m = c(H) - 1 - h = s_{\max} - m - h$  with  $h \in H$ , which implies that  $s + h = s_{\max}$ . We obtain that  $h$  must be an element  $s' \in S^*(H)$ .

Conversely let  $s + s' = s_{\max}$  with  $s' \in S^*(H)$ . Then we obtain

$$s - m = s_{\max} - m - s' = c(H) - 1 - s'$$

with  $s' \in S^*(H) \subset H$ . Hence, we get  $s - m \notin \overline{PF^*(H)}$ . □

**Corollary 1.7.** *Let  $H$  be an  $m$ -semigroup and  $s \in S^*(H)$ . Then the following are equivalent:*

- i) *We have  $s - m \in \overline{PF^*(H)}$ .*
- ii) *There exists  $s' \in S^*(H)$  such that  $s + s' = s_{\max} + \nu m$  with a positive integer  $\nu$ .  
In this case, we also have  $s' - m \in \overline{PF^*(H)}$ .*

**Proof.** Assume that  $s - m \notin \overline{PF^*(H)}$ . Then by Theorem 1.6 we have  $s + s' = s_{\max}$  for some  $s' \in S^*(H)$ . We proved that ii) implies i). The converse is true by Theorem 1.6. □

**Corollary 1.8.** *Let  $H$  be an almost symmetric  $m$ -semigroup and  $s \in S^*(H)$ . Then the following are equivalent:*

- i) *We have  $s - m \in PF^*(H)$ .*
- ii) *There exists  $s' \in S^*(H)$  such that  $s + s' = s_{\max} + m$ .  
In this case we also obtain  $s' - m \in PF^*(H)$ .*

**Proof.** Since  $H$  is almost symmetric, we have  $PF^*(H) = \overline{PF^*(H)}$ . It follows from Corollary 1.7 that ii) implies i). Assume that  $s - m \in PF^*(H)$ . By Lemma 1.1 i) and Proposition 1.4 ii) there exists  $s' \in S^*(H)$  such that

$$s - m = c(H) - 1 - (s' - m) = s_{\max} - s',$$

which implies that  $s + s' = s_{\max} + m$ . □

## 2 Numerical semigroups $H$ with $c(H) = 2g(H) - 2$ or $2g(H) - 3$

First, we investigate the set  $\overline{PF^*(H)}$  of a numerical semigroup  $H$  with  $c(H) = 2g(H) - 2$ .

**Theorem 2.1.** *Let  $H$  be an  $m$ -semigroup of genus  $g$  with  $c(H) = 2g - 2$ . Then there exists  $s \in S^*(H)$  such that*

$$\overline{PF^*(H)} = \{s - m, s_{\max} - s\} = \{s - m, 2g - 3 - (s - m)\}.$$

**Proof.** By Lemma 1.1 we may take  $s \in S^*(H)$  with  $s - m \in \overline{PH^*(H)}$ . It follows from Proposition 1.4 ii) that

$$c(H) - 1 - (s - m) = s_{\max} - s \in \overline{PH^*(H)}.$$

Then  $s - m \neq s_{\max} - m$ . Indeed, assume that  $s - m = s_{\max} - s$ . We get

$$2s = s_{\max} + m = c(H) + m - 1 + m = 2g - 2 + 2m - 1,$$

which is a contradiction. Since  $\sharp \overline{PF^*(H)} = 2g - c(H) = 2$  by Lemma 1.1 ii), we get our result. □

**Example 2.1.** Let  $H = \langle 6, 7, 10 \rangle$ . Then we obtain  $S(H) = \{6, 7, 10, 14, 17, 21\}$ , which implies that  $g(H) = 9$ ,  $s_{\max} = 21$  and  $c(H) = 16 = 2g(H) - 2$ . We have  $PF^*(H) = \{17 - 6\} = \{11\}$  and  $\overline{PF^*(H)} = \{17 - 6, 21 - 17\} = \{11, 4\}$ . In this case, we have  $t(H) = 2$ .

Since an almost symmetric numerical semigroup  $H$  satisfies  $PF^*(H) = \overline{PF^*(H)}$ , we get the following:

**Corollary 2.2.** *Let  $H$  be an almost symmetric  $m$ -semigroup of genus  $g$  with  $c(H) = 2g - 2$ . Then there exists  $s \in S^*(H)$  such that  $PF^*(H) = \{s - m, s_{\max} - s\}$ .*

**Example 2.2.** Let  $H = \langle 6, 7, 11, 16 \rangle$ . Then we obtain  $S(H) = \{6, 7, 11, 14, 16, 21\}$ , which implies that  $g(H) = 9$ ,  $s_{\max} = 21$  and  $c(H) = 16 = 2g(H) - 2$ . We have  $\overline{PF^*(H)} = PF^*(H) = \{11 - 6, 16 - 6\} = \{5, 10\}$ , which implies that  $t(H) = 3$ . Hence,  $H$  is almost symmetric.

In the case where  $c(H) = 2g(H) - 3$  a similar result to Theorem 2.1 holds as follows:

**Theorem 2.3.** *Let  $H$  be an  $m$ -semigroup of genus  $g$  with  $c(H) = 2g - 3$ . Then there exists  $\gamma \in \mathbb{N}_0 \setminus H$  such that*

$$\overline{PF^*(H)} = \{\gamma, s_{\max} - (\gamma + m), g - 2\} = \{\gamma, 2g - 4 - \gamma, g - 2\}.$$

**Proof.** We have  $\mathbb{N}_0 \setminus H \ni c(H) - 1 = 2g - 4 = 2(g - 2)$ . Hence, we get

$$\mathbb{N}_0 \setminus H \ni g - 2 = c(H) - 1 - (g - 2),$$

which implies that  $g - 2 \in \overline{PF^*(H)}$ . Let take  $\gamma \in \overline{PF^*(H)}$  which is distinct from  $g - 2$ . Then we have  $c(H) - 1 - \gamma \neq g - 2$  and  $c(H) - 1 - \gamma \neq \gamma$ . By Lemma 1.1 ii) and Proposition 1.4 ii) we get the result.  $\square$

If  $c(H) = 2g(H) - 3$ , then  $t(H) = 4$  or  $3$  or  $2$ . We will give an example in each case.

**Example 2.3.** Let  $H_1 = \langle 6, 9, 11, 14, 19 \rangle$ ,  $H_2 = \langle 6, 9, 13, 14, 17 \rangle$  and  $H_3 = \langle 6, 9, 11, 13 \rangle$ . Then we have  $g(H_i) = 10$ ,  $s_{\max} = 22$  and  $c(H_i) = 17$  for all  $i = 1, 2, 3$ .

- i)  $PF(H_1) = \{3, 8, 13, 16\}$ , hence  $t(H_1) = 4$  and  $PF^*(H_1) = \{3, 8, 13\} = \overline{PF^*(H_1)}$ .
- ii)  $PF(H_2) = \{8, 11, 16\}$ , hence  $t(H_2) = 3$  and  $PF^*(H_2) = \{8, 11\} \subset \{5, 8, 11\} = \overline{PF^*(H_2)}$ .
- iii)  $PF(H_3) = \{14, 16\}$ , hence  $t(H_3) = 2$  and  $PF^*(H_3) = \{14\} \subset \{2, 8, 14\} = \overline{PF^*(H_3)}$ .

**Theorem 2.4.** *Let  $H$  be an  $m$ -semigroup of genus  $g$  with  $c(H) = 2g - 3$ . Assume that  $g - 2 + m \in \mathbb{N}_0 \setminus H$ . Then the following hold.*

- i) *We have  $\overline{PF^*(H)} = \{g - 2 - m, g - 2, g - 2 + m\}$ .*
- ii) *We get  $PF^*(H) = \{g - 2 + m\}$ , hence  $t(H) = 2$ .*

**Proof.** We have

$$c(H) - 1 - (g - 2 + m) = 2g - 4 - (g - 2 + m) = g - 2 - m,$$

which implies that  $g - 2 - m \in \overline{PF^*(H)}$  and  $g - 2 + m \in \overline{PF^*(H)}$ , because  $g - 2 + m \in \mathbb{N}_0 \setminus H$ . Hence we get i). It follows from Lemma 1.1 i) that ii) holds, because both  $g - 2$  and  $g - 2 + m$  do not belong to  $S(H)^*$ .  $\square$

**Problem.** Let  $H$  be an  $m$ -semigroup of genus  $g$  with  $c(H) = 2g - 3$ . Assume that  $g - 2 + m \in H$ . Do we have  $t(H) \neq 2$  and  $g - 2 \in PF^*(H)$ ?

If the above statement is true, then the following hold.

- i) If  $t(H) = 3$ , then we have

$$PF^*(H) = \{g - 2, s - m\}$$

for some  $s \in S^*(H)$  with  $s \neq g - 2 + m$ .

- ii) If  $t(H) = 2$ , then we have

$$\overline{PF^*(H)} = \{g - 2 - m, g - 2, g - 2 + m\} \text{ and } PF^*(H) = \{g - 2 + m\}.$$

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