

[研究論文]

Non-Weierstrass numerical semigroups with high conductor

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Abstract

For any numerical semigroup H with conductor which is the upper bound minus 2 we give a non-Weierstrass semigroup \tilde{H} whose quotient by 4 is H . The conductor of \tilde{H} is also the upper bound minus 2.

Keywords: Numerical semigroup, Conductor, Pseudo-Frobenius number, Double covering of a curve, Non-Weierstrass semigroup

0 Introduction

Let \mathbb{N}_0 be the additive monoid of non-negative integers. A submonoid H of \mathbb{N}_0 is called a *numerical semigroup* if the complement $\mathbb{N}_0 \setminus H$ is a finite set. The cardinality of $\mathbb{N}_0 \setminus H$ is said to be the *genus* of H , which is denoted by $g(H)$. Let C be a *curve*, which means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0 in this article. For a pointed curve (C, P) of genus g we set

$$H(P) = \{h \in \mathbb{N}_0 \mid \text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = hP\}.$$

Then $H(P)$ becomes a numerical semigroup of genus g . We call $H(P)$ the *Weierstrass semigroup* of P . A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. Let t be any integer with $t \geq 2$. For any numerical semigroup \tilde{H} we define

$$d_t(\tilde{H}) = \{h \in \mathbb{N}_0 \mid th \in \tilde{H}\},$$

that is to say, $d_t(\tilde{H})$ is the quotient of \tilde{H} by t . Then $d_t(\tilde{H})$ is also a numerical semigroup. We set

$$c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteq H\},$$

which is called the *conductor* of H . It is known that $c(H) \leq 2g(H)$. A numerical semigroup H is said to be *symmetric* (resp. *quasi-symmetric*) if $c(H)$ attains the upper bound, i.e., $c(H) = 2g(H)$ (resp. the upper bound minus one, i.e., $c(H) = 2g(H) - 1$). We call $c(H) - 1$ the *Frobenius number* of H , which is denoted by $f(H)$. A non-negative integer f is called a *pseudo-Frobenius number* of H if $f \notin H$, but $f + h \in H$ for all $h \in H$ with $h > 0$. Then the Frobenius number $f(H)$ is a pseudo-Frobenius number of H . We set

$$PF(H) = \{f \in \mathbb{N}_0 \mid f \text{ is a pseudo-Frobenius number of } H\} \text{ and } PF^*(H) = PF(H) \setminus \{c(H) - 1\}$$

For a numerical semigroup H the cardinality of $PF(H)$ is called the *type* of H , which is denoted by $t(H)$. For the study of numerical semigroups which are neither symmetric nor quasi-symmetric the pseudo-Frobenius numbers of H play important roles. If $c(H) = 2g(H) - 2$, then we have $t(H) = 2$ or 3 (for example, see Proposition 1.3 i) and Theorem 2.1 in 3)).

For any non-Weierstrass numerical semigroup H Torre gives non-Weierstrass symmetric numerical semigroups \tilde{H} with $d_2(\tilde{H}) = H$ in 1) and Oliveira-Stöhr construct non-Weierstrass quasi-symmetric numerical semigroups \tilde{H} with $d_3(\tilde{H}) = H$ in 2). The following is our result:

Main Theorem. For any numerical semigroup H with $c(H) = 2g(H) - 2$ we construct non-Weierstrass numerical semigroups \tilde{H} with $d_4(\tilde{H}) = H$ such that $c(\tilde{H}) = 2g(\tilde{H}) - 2$.

We note that there are many Weierstrass numerical semigroups H with $c(H) = 2g(H) - 2$.

1 Numerical semigroups \tilde{H} with $c(d_2(\tilde{H})) = 2g(d_2(\tilde{H})) - 2$

A numerical semigroup H is said to be an m -semigroup if m is the minimum positive integer in H . In this case m is called the *multiplicity* of H , which is denoted by $m(H)$. For an m -semigroup H we set

$$s_i = \min\{h \in H \mid h \equiv i \pmod{m}\}$$

for $i = 1, \dots, m-1$ and

$$s_{max} = \max\{s_i \mid i = 1, \dots, m-1\}.$$

Then we get

$$f(H) = c(H) - 1 = s_{max} - m.$$

The set $\{m, s_1, \dots, s_{m-1}\}$ is denoted by $S(H)$, which is called the *standard basis* for H . To determine the set $PF(H)$ the following is useful.

Remark 1.1. Let H be a numerical semigroup. We set $m = m(H)$. Then we have

$$PF(H) = \{s_i - m \mid s_i + s_j \notin S(H) \text{ for all } j\}$$

(for example, see Lemma 1.1 1) in 3)).

To describe a numerical semigroup we denote by $\langle a_1, \dots, a_r \rangle$ the additive monoid generated by a_1, \dots, a_r for any positive integers a_1, \dots, a_r . We can construct numerical semigroups \tilde{H} with $c(\tilde{H}) = 2g(\tilde{H}) - 2$ from any numerical semigroup H with $c(H) = 2g(H) - 2$ as follows:

Proposition 1.2. Let H be a numerical semigroup with $c(H) = 2g(H) - 2$. Let n be an odd integer with $n \geq \max\{c(H) + m - 1, 9\}$ where we set $m = m(H)$. Take $s_i - m \in PF^*(H)$. We set $\tilde{H} = 2H + \langle n, n + 2(s_i - m) \rangle$. Then we have $2(s_i - m) \in PF^*(\tilde{H})$ and $c(\tilde{H}) = 2g(\tilde{H}) - 2$.

Proof. Since $c(H) \geq m$, it follows from the assumption $n \geq c(H) + m - 1$ that $n \geq 2m - 1$. If $n = 2m - 1$, then we have $c(H) = m$, which implies that $H = \langle m, m + 1, \dots, m + m - 1 \rangle$. In this case we obtain

$$c(H) = m = 2(m - 1) - (m - 2) = 2g(H) - (m - 2).$$

The assumption $c(H) = 2g(H) - 2$ implies that $m = 4$. In this case we have $n \geq 9 = 2m + 1$.

By the definition of \tilde{H} we have

$$S(\tilde{H}) = \{2m, 2s_1, \dots, 2s_{m-1}\} \cup \{n, n + 2s_1, \dots, n + 2s_{i-1}, n + 2(s_i - m), n + 2s_{i+1}, \dots, n + 2s_{m-1}\}.$$

By Remark 1.1 we have $2(s_i - m) \in PF^*(\tilde{H})$. Since $g(2H + \langle n \rangle) = 2g(H) + \frac{n-1}{2}$ (for example see Lemma 3.1 in 4)), we obtain $g(\tilde{H}) = 2g(H) + \frac{n-1}{2} - 1$. Hence, since $s_i \neq s_{max}$, we obtain

$$\begin{aligned} c(\tilde{H}) &= n + 2s_{max} - 2m + 1 = n + 2(s_{max} - m + 1) - 1 = n + 2c(H) - 1 \\ &= n + 2(2g(H) - 2) - 1 = 2 \left(2g(H) + \frac{n-1}{2} - 1 \right) - 2 = 2g(\tilde{H}) - 2. \end{aligned}$$

□

We can describe the set $PF^*(\tilde{H})$ in the case $t(H) = 3$.

Proposition 1.3. *Let H be a numerical semigroup with $c(H) = 2g(H) - 2$. Let n be an odd integer with $n \geq \max\{c(H) + m - 1, 9\}$ where we set $m = m(H)$. Assume that $t(H) = 3$, i.e., $PF^*(H) = \{s_i - m, s_j - m\}$ with $i \neq j$. We set $\tilde{H} = 2H + \langle n, n + 2(s_i - m) \rangle$. Then we have*

$$PF^*(\tilde{H}) = \{2s_i - 2m, n + 2(s_j - m)\}$$

Hence we get $t(\tilde{H}) = 3$.

Proof. By the description of $S(\tilde{H})$ in the proof of Proposition 1.2 we get $2s_i - 2m \in PF^*(\tilde{H})$. We will prove that $n + 2(s_j - m) \in S(\tilde{H})$. Assume that $n + 2(s_j - m) \notin S(\tilde{H})$. Then we have either

$$n + 2s_j + 2s_{j'} = n + 2s_k \text{ with some } k \neq i \text{ for some } j' \text{ or } n + 2s_j + 2s_{j'} = n + 2(s_i - m) \text{ for some } j'.$$

In the first case, we have $s_j + s_{j'} = s_k$ which contradicts $s_j - m \in PF^*(H)$. In the second case, we get $s_j + s_{j'} + m = s_i$. This is a contradiction. By Proposition 1.2 we have $c(\tilde{H}) = 2g(\tilde{H}) - 2$, which implies that $t(\tilde{H}) \leq 3$ (see Proposition 2.2 in 5)). Hence, we get our desired result. \square

2 The proof of Main Theorem

First, we will give the definition of some kind of Weierstrass numerical semigroup. A numerical semigroup H is said to be of *double covering type*, which is abbreviated to DC if there exists a double covering $\pi : \tilde{C} \rightarrow C$ of curves with a ramification point \tilde{P} such that $H(\tilde{P}) = H$. In this case it is known that $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$.

In this section let H be a numerical semigroup with $c(H) = 2g(H) - 2$. Take $s_i - m \in PF^*(H)$ where we set $m = m(H)$. Let n be an odd number with

$$n \geq \max\{s_{\max}, 4((2m - 1)(s_i - m) + 1 - g(H)) + 1\}.$$

We set

$$\tilde{H} = 2H + \langle n, n + 2(s_i - m) \rangle.$$

Assume that $(2s_i + 1, m) = 1$, i.e., $(2i + 1, m) = 1$. Let N be an odd number with $N \geq n + 2s_{\max}$. We set

$$\tilde{\tilde{H}} = 2\tilde{H} + \langle N, N + 2(2s_i - 2m) \rangle.$$

We note that $d_2(\tilde{\tilde{H}}) = \tilde{H}$, $d_2(\tilde{H}) = H$ and $d_4(\tilde{\tilde{H}}) = H$.

First, we want to show that $\tilde{\tilde{H}}$ is not DC. For that purpose we will prove the following: if $\tilde{\tilde{H}}$ were DC, then we would lead a contradiction. We assume that $\tilde{\tilde{H}}$ is DC. Then we have

$$d_2(\tilde{\tilde{H}}) = \tilde{H} = 2H + \langle n, n + 2(s_i - m) \rangle,$$

which is Weierstrass, i.e., there exists a pointed curve (C, P) with $H(P) = \tilde{H}$. By Proposition 1.2 we have $c(\tilde{H}) = 2g(\tilde{H}) - 2$. Hence, we get

$$K_C \sim (2g(\tilde{H}) - 4)P + Q_1 + Q_2, Q_i \neq P \text{ for } i = 1, 2.$$

Since $\tilde{\tilde{H}}$ is DC, there is a double covering

$$\pi : \tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \mathcal{O}_C(-D)) \rightarrow C$$

with a ramification \tilde{P} over P such that $H(\tilde{P}) = \tilde{\tilde{H}}$ where D is a divisor on C such that $2D$ is linearly equivalent to a reduced divisor $P + P_2 + P_3 + \cdots + P_{N-1}$ (refer to the fifth line of page 547 from the fifth line from bottom of page 546 in 6)). Moreover, we have that D is linearly equivalent to $\frac{N+1}{2}P - Q$ for some $Q \in C$ with $Q \neq P$. Indeed, since N is odd, $N - 1$ is even. Hence, by Proposition 2.1 in 7) we have

$$h^0((N-1)\tilde{P}) = h^0\left(\left(\frac{N-1}{2}\right)P\right) + h^0\left(\left(\frac{N-1}{2}\right)P - D\right)$$

and

$$h^0((N+1)\tilde{P}) = h^0\left(\left(\frac{N+1}{2}\right)P\right) + h^0\left(\left(\frac{N+1}{2}\right)P - D\right).$$

We note that $N \in H(\tilde{P})$ and that $N+1 \in H(\tilde{P})$ if and only if

$$h^0\left(\left(\frac{N-1}{2}\right)P\right) + 1 = h^0\left(\left(\frac{N+1}{2}\right)P\right),$$

because $d_2(H(\tilde{P})) = d_2(\tilde{H}) = \tilde{H} = H(P)$. Since N is the minimum odd integer in $H(\tilde{P})$, we have that

$$h^0\left(\left(\frac{N-1}{2}\right)P - D\right) = 0$$

and

$$h^0\left(\left(\frac{N+1}{2}\right)P - D\right) = 1.$$

Hence, we have that $\left(\frac{N+1}{2}\right)P - D$ is linearly equivalent to a point Q different from P . Moreover, we have

$$h^0((N+2(2s_i-2m)-1)\tilde{P}) = h^0\left(\left(\frac{N+2(2s_i-2m)-1}{2}\right)P\right) + h^0((2s_i-2m-1)P+Q)$$

and

$$h^0((N+2(2s_i-2m)+1)\tilde{P}) = h^0\left(\left(\frac{N+2(2s_i-2m)+1}{2}\right)P\right) + h^0((2s_i-2m)P+Q).$$

We note that $N+2(2s_i-2m) \in H(\tilde{P})$ and that $N+2(2s_i-2m)+1 \in H(\tilde{P})$ if and only if

$$h^0\left(\left(\frac{N+2(2s_i-2m)-1}{2}\right)P\right) + 1 = h^0\left(\left(\frac{N+2(2s_i-2m)+1}{2}\right)P\right).$$

Hence, we obtain that

$$h^0((2s_i-2m)P+Q) = h^0((2s_i-2m-1)P+Q) + 1.$$

Thus, there is a rational function f on C such that its polar divisor $(f)_\infty$ is equal to $(2s_i-2m)P+Q$. Moreover, $H(P) = \tilde{H} \ni 2m$ implies that there exists a rational function f_1 on C such that $(f_1)_\infty = 2mP$. Let ν be the degree of the field extension $k(C)/k(f, f_1)$. Then $2s_i-2m+1$ and $2m$ are divisible by ν . Hence, $2m$ and $2s_i+1$ are divisible by ν . By the assumption $(m, 2s_i+1) = 1$ we have $\nu = 1$, i.e., $k(C) = k(f, f_1)$. Hence, we get

$$2g(H) + \frac{n-1}{2} - 1 = g(\tilde{H}) = g(C) \leq (\deg f - 1)(\deg f_1 - 1) = (2s_i-2m)(2m-1)$$

(for example, see Theorem 1.1 in 8)), which implies that $n \leq 4((2m-1)(s_i-m) + 1 - g(H)) - 1$. This contradicts the assumption on n . Hence, \tilde{H} is not DC.

If $N \geq 16g(H) + 4n - 9$, then

$$\begin{aligned} g(\tilde{H}) - 6g(\tilde{H}) &= 2g(\tilde{H}) + \frac{N-1}{2} - 1 - 6g(\tilde{H}) \geq -4g(\tilde{H}) + 8g(H) + 2n - 5 - 1 \\ &= -4\left(2g(H) + \frac{n-1}{2} - 1\right) + 8g(H) + 2n - 6 = 0. \end{aligned}$$

By 1) and 9) \tilde{H} is a non-Weierstrass numerical semigroup with $c(\tilde{H}) = 2g(\tilde{H}) - 2$ and $d_4(\tilde{H}) = H$.

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