[研究論文]

Weierstrass semigroups on double covers of plane curves of degree 7

Seon Jeong KIM[1], Jiryo KOMEDA[2]

1 Department of Mathematics and RINS, Gyeongsang National University

2 Center for Basic Education and Integrated Learning, Kanagawa Institute of Technology

Abstract

We investigate Weierstrass semigroups of ramification points on double covers of plane curves of degree 7. We treat the cases where the Weierstrass semigroups are generated by at most 5 elements and the ramification point is on a total flex.

Keywords: Numerical semigroup, Weierstrass semigroup, Plane curve, Double cover of a curve

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is a finite set. The cardinality of $\mathbb{N}_0 \backslash H$ is said to be the *genus* of H, which is denoted by $g(H)$. Let C be a *curve*, which means a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0 in this article. For a pointed curve (C, P) of genus g we set

$$H(P) = \{h \in \mathbb{N}_0 \mid \text{ there is a rational function } f \text{ on } C \text{ such that } (f)_\infty = hP\}$$

where $(f)_\infty$ is the polar divisor of the function f. Then $H(P)$ becomes a numerical semigroup of genus g. We call $H(P)$ the *Weierstrass semigroup* of P. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. For any numerical semigroup H we define

$$d_2(H) = \{h \in \mathbb{N}_0 \mid 2h \in H\},$$

that is to say, $d_2(H)$ is the quotient of H by 2. Then $d_2(H)$ is also a numerical semigroup. If $\pi : \tilde{C} \longrightarrow C$ is a double cover of a curve with a ramification point $\tilde{P}$ over P, then we have $d_2(H(\tilde{P})) = H(P)$. Such a numerical semigroup $H = H(\tilde{P})$ is said to be *of double covering type*.

Let C be a smooth plane curve of degree $d \geqq 4$ and P its total flex, i.e., $\mathrm{ord}_P\, C.T_P = d$ where T_P is the tangent line at P on C and $\mathrm{ord}_P\, C.T_P$ is the the multiplicity at P of the intersection divisor $C.T_P$ of C with T_P. Then we have $H(P) = \langle d-1, d\rangle$ where for any positive integers $a_1, \ldots, a_n$ we denote by $\langle a_1, \ldots, a_n\rangle$ the additive monoid generated by $a_1, \ldots, a_n$. Conversely, if (C, P) is a pointed curve with $H(P) = \langle d-1, d\rangle$, $d \geqq 3$, then C is a plane curve with total flex P. In this article we are interested in the double covers $\pi : \tilde{C} \longrightarrow C$ of curves with ramification points on the points whose Weierstrass semigroups are $\langle d-1, d\rangle$. We pose the following problem:

TF Hurwitz Problem. Let d be a positive integer with $d \geqq 3$. Let H be any numerical semigroup with $d_2(H) = \langle d-1, d\rangle$ with $g(H) \geqq \dfrac{3(d-2)(d-1)}{2}$. Then is H *of double covering type*?

In the above problem TF means total flexes. Under the assumption $g(H) \geqq \dfrac{3(d-2)(d-1)}{2}$ we can construct a double cover $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P for a pointed plane curve (C, P) with $H(P) = \langle d-1, d\rangle$. But we cannot prove that $H(\tilde{P}) = H$. TF Hurwitz Problem was solved for $d \leqq 6$. For $d = 3$ the result is classical (for example, see Theorem 3.5 in 1)). If $d = 4$, this problem was solved[2)]. In the cases $d = 5$ and 6 the problem are proved in 3)and 4) respectively. We treat the case $d = 7$ in this article. Let n be the minimum odd integer in H. Then we have $g(H) = (d-1)(d-2) + \dfrac{n-1}{2} - r$ with a non-negative integer r [5)].

Main Theorem. Let H be a numerical semigroup with $d_2(H) = \langle 6, 7\rangle$ and $g(H) \geqq 45$.
i) If H is generated by 5 elements and $r \leqq 6$, then it is *of double covering type*.
ii) If H is generated by 4 elements and $H \neq 2\langle 6, 7\rangle + \langle n, n+8\rangle$, then it is *of double covering type*.

2 The classification of H with $r \leqq 6$ generated by at most 5 elements

A numerical semigroup H is said to be an m-*semigroup* if m is the minimum positive integer in H. In this case m is called the *multiplicity* of H, which is denoted by $m(H)$. For an m-semigroup H we set

$$s_i = \min\{h \in H \mid h \equiv i \bmod m\}$$

for $i = 1, \dots, m-1$. The set $\{m, s_1, \dots, s_{m-1}\}$ is denoted by $S(H)$, which is called the *standard basis* for H.

From now on, let H be a numerical semigroup with $d_2(H) = H_7$ where we set $H_7 = \langle 6, 7\rangle$. We set

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

We assume $n \geqq 35$. Then we have $2\langle 6, 7\rangle + n\mathbb{N}_0 \subseteqq H$. We note that

$$S(2\langle 6,7\rangle + n\mathbb{N}_0) = \{12, 14, 28, 42, 56, 70\} \cup \{n, n+14, n+28, n+42, n+56, n+70\}.$$

We associate to H the diagram where $\odot$, $\circ$ and $\times$ indicate an integer which is in $M(H)$, $H\backslash M(H)$ and $\mathbb{N}_0\backslash H$ respectively. Here $M(H)$ denotes the minimal set of generators for the monoid H. Let $r = r(H)$ be the number of $\odot$ and $\circ$. Then $0 \leqq r \leqq 15$. Moreover, we obtain $g(H) = 30 + \dfrac{n-1}{2} - r$. Let $t(H)$ be the cardinality of the set

$$\{u \in M(H) \mid u \text{ is an odd integer distinct from } n\}.$$

For example, we associate the following diagram with the numerical semigroup $H = 2\langle 6, 7\rangle + \langle n, n+16, n+32\rangle$:

$$\begin{array}{cccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \times \\
(n) & \bullet & \odot & \times & \times & \times \downarrow +12 \\
\searrow +14 & (n+14) & \bullet & \circ & \odot & \times \\
 & & (n+28) & \bullet & \circ & \circ \\
 & & & (n+42) & \bullet & \circ \\
 & & & & (n+56) & \bullet \\
 & & & & & (n+70)
\end{array}$$

In this case we have $r = r(H) = 6$, $g(H) = 30 + \dfrac{n-1}{2} - 6$ and $t(H) = 2$.

From here we list numerical semigroups H with $d_2(H) = \langle 6, 7\rangle$, $r = r(H) \leqq 6$ and $t(H) = 1$ or 2. We consider the numerical semigroups with diagrams such that there are no $\odot$'s in a left column of the column with $\odot$ in the diagram below.

(1) Consider

$$\begin{array}{cccccc}
\rightarrow +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \odot \\
(n) & \bullet & \times & \times & \times & \circ \downarrow +12 \\
\searrow +14 & (n+14) & \bullet & \times & \times & \circ \\
 & & (n+28) & \bullet & \times & \circ \\
 & & & (n+42) & \bullet & \circ \\
 & & & & (n+56) & \bullet \\
 & & & & & (n+70)
\end{array}$$

Then we have $H = 2H_7 + \langle n, n+2t_1 \rangle$ with $t_1 = 35 - 6l$ where l is a positive integer with $l \leqq 5$.

(2) Consider

$$\begin{array}{cccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \times \\
(n) & \bullet & \times & \times & \odot & \times \downarrow +12 \\
\searrow +14 & (n+14) & \bullet & \times & \circ & \circ \\
 & & (n+28) & \bullet & \circ & \circ \\
 & & & (n+42) & \bullet & \circ \\
 & & & & (n+56) & \bullet \\
 & & & & & (n+70)
\end{array}$$

Then we have $H = 2H_7 + \langle n, n+2t_1 \rangle$ with $t_1 = 28 - 6l$ where l is a positive integer with $l \leqq 3$, $H = 2H_7 + \langle n, n+2(28-12), n+2t_2 \rangle$ where $t_2 = 35 - 6l$ with $l = 3, 4$ and $H = 2H_7 + \langle n, n+2(28-6), n+2t_2 \rangle$ where $t_2 = 35 - 6l$ with $2 \leqq l \leqq 5$.

(3) Consider

$$\begin{array}{cccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \times \\
(n) & \bullet & \times & \odot & \times & \times \downarrow +12 \\
\searrow +14 & (n+14) & \bullet & \circ & \circ & \times \\
 & & (n+28) & \bullet & \circ & \circ \\
 & & & (n+42) & \bullet & \circ \\
 & & & & (n+56) & \bullet \\
 & & & & & (n+70)
\end{array}$$

Then we have $H = 2H_7 + \langle n, n+2t_1 \rangle$ with $t_1 = 21 - 6l$ where l is a positive integer with $l \leqq 2$, $H = 2H_7 + \langle n, n+2(21-6), n+2(28-12) \rangle$ and $H = 2H_7 + \langle n, n+2(21-6), n+2t_2 \rangle$ where $t_2 = 35 - 6l$ with $l = 2, 3$.

(4) Consider

$$\begin{array}{cccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \times & \times & \times & \times & \times \\
(n) & \bullet & \odot & \times & \times & \times \downarrow +12 \\
\searrow +14 & (n+14) & \bullet & \circ & \times & \times \\
 & & (n+28) & \bullet & \circ & \times \\
 & & & (n+42) & \bullet & \circ \\
 & & & & (n+56) & \bullet \\
 & & & & & (n+70)
\end{array}$$

Then we have $H = 2H_7 + \langle n, n+2(14-6) \rangle$, $H = 2H_7 + \langle n, n+2(14-6), n+2(28-12) \rangle$ and $H = 2H_7 + \langle n, n+2(14-6), n+2t_2 \rangle$ where $t_2 = 35 - 6l$ with $l = 2, 3$.

(5) Consider

$$\begin{array}{cccccc}
\to +2 & (n+2) & (n+4) & (n+6) & (n+8) & (n+10) \\
\bullet & \odot & \times & \times & \times & \times \\
(n) & \bullet & \circ & \times & \times & \times \downarrow +12 \\
\searrow +14 & (n+14) & \bullet & \circ & \times & \times \\
 & & (n+28) & \bullet & \circ & \times \\
 & & & (n+42) & \bullet & \circ \\
 & & & & (n+56) & \bullet \\
 & & & & & (n+70)
\end{array}$$

Then we have $H = 2H_7 + \langle n, n+2(7-6) \rangle$, and $H = 2H_7 + \langle n, n+2(7-6), n+2(35-12) \rangle$.

3 The case where H with $r \leqq 6$ is generated by 5 elements

By Theorem 2.5 in 3) we know that the following numerical semigroups H with $d_2(H) = \langle 6, 7\rangle$ are *of double covering type*.

Theorem 3.1 *Let n be an odd number with $n \geqq 35$. Let H be a numerical semigroup with $d_2(H) = H_7 = \langle 6, 7\rangle$ which is one of the following type:*

(i) $2H_7 + \langle n, n+2(35-12), n+2t_2\rangle$ *with* $t_2 = 7(7-m)-6$ *where m is an integer with* $3 \leqq m \leqq 6$ *and* $n \geqq (7-1)(7-2)+1+2m$.

(ii) $2H_7 + \langle n, n+2(35-6l), n+2(28-6)\rangle$ *where l is an integer with* $3 \leqq l \leqq 5$ *and* $n \geqq (7-1)(7-2)+3+2l$.

(iii) $2H_7 + \langle n, n+2(21-6), n+2(35-18)\rangle$ *with* $n \geqq (7-1)(7-2)+11$.

(iv) $2H_7 + \langle n, n+2(28-12), n+2(35-18)\rangle$ *with* $n \geqq (7-1)(7-2)+11$.

(v) $2H_7 + \langle n, n+2(21-6), n+2(28-12)\rangle$ *with* $n \geqq (7-1)(7-2)+11$.

Then H is of double covering type.

In this section we consider the case where $t(H) = 2$, i.e., H is generated by 5 elements.

The case (2) in section 2. By Theorem 3.1 (i), (ii) and (iv), any H with $t(H) = 2$ except

$$H = 2H_7 + \langle n, n+2(28-12), n+2(35-24)\rangle$$

is *of double covering type*.

The case (3) in section 2. By Theorem 3.1 (i), (iii) and (v), any H with $t(H) = 2$ except

$$H = 2H_7 + \langle n, n+2(21-6), n+2(35-24)\rangle$$

is *of double covering type*.

The case (4) in section 2. By Theorem 3.1, H with $t(H) = 2$ which is neither

$$2H_7 + \langle n, n+2(14-6), n+2(28-12)\rangle \text{ nor } 2H_7 + \langle n, n+2(14-6), n+2(35-18)\rangle$$

is *of double covering type*.

The case (5) in section 2. By Theorem 3.1 (i), $H = 2H_7 + \langle n, n+2(7-6), n+2(35-12)\rangle$ is *of double covering type*

Theorem 3.2 *Let C be a non-singular plane curve of degree $d \geqq 4$. Let E be an effective divisor of degree $d-1$ on C. We set $E = Q_1 + \cdots + Q_{d-1}$ where Q_i's are points of C. Then we have $h^0(E) = 2$ if and only if $Q_1, \ldots, Q_{d-1}$ lie on a line* [6].

Theorem 3.3 *Let (C, P) be a pointed non-singular plane curve of degree 7 and H a numerical semigroup with $d_2(H) = H(P)$ and $g(H) \geqq 45$. Set*

$$n = \min\{h \in H \mid h \text{ is odd}\}.$$

We note that

$$g(H) = 30 + \frac{n-1}{2} - r$$

with some non-negative integer r. Let $Q_1, \ldots, Q_r$ be points of C different from P with $h^0(Q_1+\cdots+Q_r) = 1$. Moreover, assume that H has an expression

$$H = 2d_2(H) + \langle n, n+2l_1, \ldots, n+2l_s\rangle$$

with positive integers $l_1, \ldots, l_s$ such that for any curve C_4 of degree 4 the inequality $C_4.C \geqq (l_i-1)P + Q_1 + \cdots + Q_r$ implies that $C_4.C \geqq l_iP + Q_1 + \cdots + Q_r$, i.e.,

$$h^0(K-(l_i-1)P-Q_1-\cdots-Q_r) = h^0(K-l_iP-Q_1-\cdots-Q_r)$$

where K is a canonical divisor on C. Then there is a double cover $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$, i.e., H is of double covering type.

Proof. We consider the divisor

$$D = \frac{n+1}{2}P - (Q_1 + \cdots + Q_r).$$

By the assumption $g(H) \geqq 45$ we have

$$\deg(2D - P) = n - 2r = 2g - 59 \geqq 90 - 59 = 31 = 2g(C) + 1$$

where $g(C)$ is the genus of the plane curve C of degree 7. Hence, the complete linear system $|2D - P|$ is base-point free. By Theorem 2.2 in 5) we can construct a double cover

$$\pi : \tilde{C} = \text{Spec}\ (\mathcal{O}_C \oplus \mathcal{O}(-D)) \longrightarrow C$$

with a ramification point $\tilde{P}$ over P with $H(\tilde{P}) = H$. □

Hereafter, let C be a non-singular plane curve of degree 7 with a total flex P and $Q_1, \ldots, Q_r$ be points of C distinct from P. We set $E_r = Q_1 + \cdots + Q_r$.

Theorem 3.4 $H = 2H_7 + \langle n, n + 2(28-12), n + 2(35-24)\rangle$ *is of double covering type.*

Proof. In this case $r = 6$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line L_1 with $Q_5 \notin L_1$ and $Q_6 \notin L_1$. By Theorem 3.2 we obtain $h^0(Q_1 + \cdots + Q_6) = 1$. Let C_4 be a curve of degree 4 with $C_4.C \geqq 10P + E_6$. Since $C.T_P^2 L_1 \geqq 14P + Q_1 + Q_2 + Q_3 + Q_4$, by Bézout's Theorem (see Theorem p.172 in 7)) we must have $C_4 = T_P^2 L_1 L_2$ with a line L_2, which implies that $C_4.C \geqq 14P + E_6$. Thus, we get $h^0(K - 10P - E_6) = h^0(K - 11P - E_6) = h^0(K - 14P - E_6) = 1$. Moreover, we have $h^0(K - 15P - E_6) = 0$. By Theorem 3.3 H is *of double covering type.* □

Theorem 3.5 $H = 2H_7 + \langle n, n + 2(21-6), n + 2(35-24)\rangle$ *is of double covering type.*

Proof. In this case $r = 6$. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line L_1 with $Q_5 \notin L_1$ and $Q_6 \notin L_1$. Let us take a line L_P which is distinct from T_P. Let Q_5 and Q_6 be on the line L_P. Let C_4 be a curve of degree 4 with $C_4.C \geqq 10P + E_6$. We obtain $C_4 = T_P^2 L_1 L_P$. Hence, we have

$$h^0(K - 10P - E_6) = h^0(K - 11P - E_6) = h^0(K - 14P - E_6) = h^0(K - 15P - E_6) = 1.$$

By Theorem 3.3 H is *of double covering type.*

Theorem 3.6 $H = 2H_7 + \langle n, n + 2(14-6), n + 2(35-18)\rangle$ *is of double covering type.*

Proof. In this case $r = 6$. Let L_P be a line through P which is distinct from T_P. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line L_P. Let Q_5 and Q_6 be points such that the line L_0 through the two points does not contain P. Let C_4 be a curve of degree 4 with $C_4.C \geqq 7P + E_6$. Then we have $C_4 = T_P L_P C_2$ where C_2 is a conic containing Q_5 and Q_6. Hence we get $h^0(K - 7P - E_6) = h^0(K - 8P - E_6)$. Moreover, let C_4' be a curve of degree 4 with $C_4'.C \geqq 16P + E_6$. Then we should have $C_4' = T_P^2 L_P L_0$, which implies that $\text{ord}_P(C_4'.C) = 15$. This is a contradiction. Hence, we get $h^0(K - 16P - E_6) = 0$. Thus, H is *of double covering type.* □

Theorem 3.7 $H = 2H_7 + \langle n, n + 2(14-6), n + 2(28-12)\rangle$ *is of double covering type.*

Proof. In this case $r = 6$. Let L_P and L_P' be distinct lines through P different from T_P. Let us take $Q_1, \ldots, Q_4$ such that the four points lie on the line L_P. Let us take Q_5 and Q_6 such that the two points lie on the line L_P' . Let C_4 be a curve of degree 4 with $C_4.C \geqq 7P + E_6$. Then we have $C_4 = T_P L_P C_2$ where C_2 is a conic containing Q_5 and Q_6. Hence we get $h^0(K - 7P - E_6) = h^0(K - 8P - E_6)$. Moreover, let C_4' be a curve of degree 4 with $C_4'.C \geqq 15P + E_6$. Then we should have $C_4' = T_P^2 L_P L_P'$, which implies that $h^0(K - 15P - E_6) = h^0(K - 16P - E_6) = 1$. Thus, H is *of double covering type.* □

4 The case where H is generated by 4 elements

In this section we treat the numerical semigroups H with $d_2(H) = \langle 6, 7\rangle$ and $t(H) = 1$. By Theorem 2.5 in 3) we know that the following numerical semigroups H with $d_2(H) = \langle 6, 7\rangle$ are *of double covering type*.

Theorem 4.1 *Let n be an odd number with $n \geqq 35$. Let H be a numerical semigroup which is one of the following:*

(i) $2H_7 + \langle n, n+2t_1\rangle$ *with* $t_1 = 35 - l(7-1)$ *where l is a positive integer with $l \leqq 5$ and* $n \geqq (7-1)(7-2)+1+2l$.

(ii) $2H_7 + \langle n, n+2t_1\rangle$ *with* $t_1 = s_{7-m} - (7-1)$ *where m is an integer with $3 \leqq m \leqq 6$ and* $n \geqq (7-1)(7-2)-1+2m$.

(iii) $2H_7 + \langle n, n+2t_1\rangle$ *with* $t_1 = s_{7-m} - 2(7-1)$ *where m is an integer with $3 \leqq m \leqq 5$ and* $n \geqq (7-1)(7-2)-3+4m$.

Then H is of double covering type.

With Theorem 4.1, we cannot say that the following three semigroups H with $d_2(H) = \langle 6, 7\rangle$ and $t(H) = 1$ are *of double covering type* or not.

(1) $2H_7 + \langle n, n+20\rangle$ (2) $2H_7 + \langle n, n+8\rangle$ (3) $2H_7 + \langle n, n+6\rangle$.

To prove that the numerical semigroups in (1) and (3) are *of double covering type* we need the following:

Theorem 4.2 (Cayley-Bacharach) (For example, see p. 671 in 7)) *Let C be a non-singular plane curve. Let X_1 and X_2 be two plane curves of degree d and e respectively, meeting in a collection Γ of de points of C with multiplicity. Let Y be a curve of degree $d+e-3$ such that the intersection $Y.C$ contains all but one point of Γ. Then $Y.C$ contains that remaining point also.*

For the case (1) we use the following curve:

Lemma 4.3 *The plane curve of degree 7 defined by the equation*

$$(yz^2 - x^3)\left(\frac{1}{2}z^4 + ax^4\right) + (yz^2 + x^3 - 2y^3)\left(\frac{1}{2}z^4 + by^4\right) = 0$$

is nonsingular for general a and b.

Proof. We have

$$(yz^2 - x^3)\left(\frac{1}{2}z^4 + ax^4\right) + (yz^2 + x^3 - 2y^3)\left(\frac{1}{2}z^4 + by^4\right) = z^4\left(yz^2 - y^3\right) + ax^4(yz^2 - x^3) + by^4(yz^2 + x^3 - 2y^3) = F.$$

We will calculate the base locus, i.e., the intersection of the three curves

$$z^4(yz^2 - y^3) = 0,\ x^4(yz^2 - x^3) = 0 \text{ and } y^4(yz^2 + x^3 - 2y^3) = 0.$$

If $z = 0$, then we have $x = 0$ and $y = 0$. This is a contradiction. Hence, we may set $z = 1$. Thus, we consider the intersection of the following three curves

$$y - y^3 = 0,\ x^4(y - x^3) = 0 \text{ and } y^4(y + x^3 - 2y^3) = 0.$$

Hence, we have $y = 0$, 1 or -1. Let $y = 0$. Then we have $x = 0$. Hence, we obtain the point $(0:0:1)$. Let $y = 1$. Then $x = 1, \omega$ or ω^2 where ω is a primitive cubic root of unity. Hence, we get the three points $(1:1:1)$, $(\omega:1:1)$ and $(\omega^2:1:1)$. Let $y = -1$. Then $x = -1, -\omega$ or $-\omega^2$. Thus, we get the three points $(-1:-1:1)$, $(-\omega:-1:1)$ and $(-\omega^2:-1:1)$. Therefore, the base locus consists of the seven points. The partial differentials of F are the following:

$$F_x = 4ax^3(yz^2 - x^3) - 3ax^6 + 3bx^2y^4 = ax^3(4yz^2 - 7x^3) + 3bx^2y^4$$

$$F_y = z^6 - 3y^2z^4 + ax^4z^2 + 4by^3(yz^2 + x^3 - 2y^3) + by^4(z^2 - 6y^2) = z^6 - 3y^2z^4 + ax^4z^2 + by^3(5yz^2 + 4x^3 - 14y^2)$$

$$\text{and } F_z = 4z^3(yz^2 - y^3) + 2ax^4yz + 2by^5z.$$

For general a and b we have

$$F_x(0,0,1)=1\neq 0,\ F_x(1,1,1)=-3a+3b\neq 0,\ F_x(\omega,1:1)=-3a+3b\omega^2\neq 0,\ F_x(\omega^2,1:1)=-3a+3b\omega\neq 0,$$

$$F_x(-1,-1,1)=3a+3b\neq 0,\ F_x(-\omega,-1,1)=-3a+3b\omega^2\neq 0 \text{ and } F_x(-\omega^2,-1,1)=-3a+3b\omega\neq 0.$$

Hence, the plane curve defined by $F=0$ is non-singular for general a and b by Bertini's theorem (for example, see p.137 in 7)). □

Theorem 4.4 *Let n be an odd number with $n \geqq 43$. Then $2H_7+\langle n, n+20\rangle$ is of double covering type.*

Proof. In this case $r=6$. Let C be the non-singular plane curve of degree 7 in Lemma 4.3. We set $P=(0:0:1)$. Then we have $C.T_P=7P$, in this case T_P is the line defined by $y=0$. Let C_{31} and C_{32} be the cubics defined by the equations $yz^2-x^3=0$ and $yz^2+x^3-2y^3=0$, respectively. Then the intersection $C_{31}.C_{32}$ of C_{31} and C_{32} is $3P+\sum_{i=1}^{6}Q_i$ where $Q_1=(1:1:1)$, $Q_2=(1:1:\omega)$, $Q_3=(1:1:\omega^2)$, $Q_4=(1:-1:-1)$, $Q_5=(1:-1:-\omega)$ and $Q_6=(1:-1:-\omega^2)$. Since the six points $Q_1,\ldots,Q_6$ are not on a line. Hence by Theorem 3.2 we get $h^0(Q_1,\ldots,Q_6)=1$. Let C_4 be a curve of degree 4 with $C_4.C \geqq 9P+E_6$. Then we obtain $C_4=T_PC_3$ where C_3 is a cubic. Since $C_3.C\geqq 2P+\sum_{i=1}^{6}Q_i$ and $C_{31}.C_{32}=3P+\sum_{i=1}^{6}Q_i$, by Theorem 4.2 we obtain $C_3.C\geqq 3P+\sum_{i=1}^{6}Q_i$. Hence we get $C_4.C\geqq 10P+E_6$. By Theorem 3.3 the numerical semigroup $2H_7+\langle n, n+20\rangle$ is *of double covering type*. □

Lemma 4.5 *The plane curve of degree 7 defined by the equation*

$$(yz^2-x^3)\left(\frac{1}{2}z^4+ax^4\right)+(yz^3+x^3z-2y^4)\left(\frac{1}{2}z^3+by^3\right)=0$$

is nonsingular for general a and b.

Proof. We have

$$(yz^2-x^3)\left(\frac{1}{2}z^4+ax^4\right)+(yz^3+x^3z-2y^4)\left(\frac{1}{2}z^3+by^3\right)=yz^6-y^4z^3+ax^4(yz^2-x^3)+by^3(yz^3+x^3z-2y^4)=F.$$

The base locus is the intersection of

$$z^3(yz^3-y^4)=0,\ x^4(yz^2-x^3)=0 \text{ and } y^3(yz^3+x^3z-2y^4)=0.$$

If $z=0$, then we have $x=0$ and $y=0$. This is a contradiction. Hence, we may set $z=1$. Thus, we consider the intersection of the following three curves

$$y-y^4=0,\ x^4(y-x^3)=0 \text{ and } y^3(y+x^3-2y^4)=0.$$

Since we have $y-y^4=y(1-y^3)=0$, we obtain $y=0$, $y=1$, $y=\omega$ od $y=\omega^2$. If $y=0$, then $x=0$. Hence, we get the point $(0:0:1)$. If $y=1$, then $x^3-1=0$. Thus, we have the three points $(1:1:1)$, $(\omega:1:1)$ and $(\omega^2:1:1)$. If $y=\omega$, then we obtain the three points $(\zeta:\omega:1)$, $(\zeta^4:\omega,1)$ and $(\zeta^7:\omega:1)$ where ζ is a primitive 9-th root of unity. If $y=\omega^2$, then we obtain the three points $(\zeta^2:\omega^2:1)$, $(\zeta^5:\omega^2:1)$ and $(\zeta^8:\omega^2:1)$. The partial differentials of F are the following:

$$F_x=ax^3(4yz^2-7x^3)+3bx^2y^3z,\ F_y=z^6-4y^3z^3+ax^4z^2+by^2(3x^3z+4yz^3-14y^4)$$

$$\text{and } F_z=6yz^5-3y^4z^2+2ax^4yz+by^3(3yz^2+x^3).$$

Hence, we have

$$F_y(0,0,1)=1\neq 0,\ F_x(1,1,1)=-3a+3b\neq 0$$

for general a and b. For the remaining eight points the values of the function F_x are not zero for general a and b. Hence the plane curve of degree 7 is non-singular. □

Theorem 4.6 *Let n be an odd number with $n \geqq 49$. Then $2H_7 + \langle n, n+6 \rangle$ is of double covering type.*

Proof. In this case we have $r = 9$. Let C be the non-singular plane curve of degree 7 in Lemma 4.5. We set $P = (0:0:1)$. Then we have $C.T_P = 7P$, in this case T_P is the line defined by $y = 0$. Let C_{31} be the cubic defined by the equation $yz^2 - x^3 = 0$ and C_{41} be the quartic defined by the equation $yz^3 + x^3z - 2y^4 = 0$. We may assume that $z = 1$. Hence we consider the equations $y - x^3 = 0$ and $y + x^3 - 2y^4 = 0$, which imply that $x^3(x^9 - 1) = 0$. Let η be a primitive 9-th root of unity. We set $Q_l = (\eta^l : \eta^{3l} : 1)$ for $l = 0, 1, 2, \ldots, 8$. Then the intersection divisor $C_{31}.C_{41}$ is $3P + \sum_{l=0}^{8} Q_l$. Let C_4 be a curve of degree 4 with $C_4.C \geqq 2P + E_9$. Then by Theorem 4.2 we get $C_3.C \geqq 3P + \sum_{i=0}^{8} Q_i$. We want to show that $h^0(K - E_9) = 6$. Let C_4 be a curve of degree 4 with $C_4.C \geqq E_9$, i.e., it is defined by the equation

$$F_4(x,y,z) = c_{400}x^4 + c_{310}x^3y + c_{301}x^3z + c_{220}x^2y^2 + c_{211}x^2yz + c_{202}x^2z^2 + c_{130}xy^3 +$$

$$c_{121}xy^2z + c_{112}xyz^2 + c_{103}xz^3 + c_{040}y^4 + c_{031}y^3z + c_{022}y^2z^2 + c_{013}yz^3 + c_{004}z^4 = 0$$

satisfying $F_4(\eta^l, \eta^{3l}, 1) = 0$ for $l = 0, 1, \ldots, 8$. The rank of the matrix of the coefficients of the system of linear equations $F_4(\eta^l, \eta^{3l}, 1) = 0$ ($l = 0, 1, \ldots, 8$) with 15 variables c_{ijk}, $i + j + k = 4$ is 9, because some 9 by 9 minor of the matrix is Vandermonde's determinant. Hence, we get $h^0(K - E_9) = 15 - 9 = 6$, which implies that $h^0(E_9) = 1$. Thus, $2H_7 + \langle n, n+6 \rangle$ is *of double covering type*. □

We do not know whether the remaining numerical semigroup $2H_7 + \langle n, n+8 \rangle$ is *of double covering type* or not.

Acknowledgment. This work was supported by JSPS KAKENHI Grant Number18K03228.

References

1) J. Komeda, *A numerical semigroup from which the semigroup gained by dividing by two is either $\mathbb{N}_0$ or a 2-semigroup or $\langle 3, 4, 5 \rangle$*, Research Reports of Kanagawa Institute of Technology **B-33**, 37–42 (2009).
2) J. Komeda, *On Weierstrass semigroups of double coverings of genus three curves*, Semigroup Forum **83** (2011), 479–488.
3) S.J. Kim and J. Komeda, *Weierstrass semigroups on double covers of plane curves of degree* 5, Kodai Math. J.**38** (2015), 270–288.
4) S.J. Kim and J. Komeda, *Weierstrass semigroups on double covers of plane curves of degree six with total flexes*, Bulletin of the Korean Mathematical Society **55** (2018), 611-624.
5) J. Komeda and A. Ohbuchi, *On double coverings of a pointed non-singular curve with any Weierstrass semigroup*, Tsukuba J. Math. Soc. **31** (2007) 205-215.
6) M. Namba, Families of meromorphic functions on compact Riemann surfaces, Lecture Notes in Math. **767**, Springer-Verlag, Berlin, 1979.
7) P. Griffiths and J. Harris, Principles of algebraic geometry, John Wiley & Sons, Inc.,1978.