The Optimal Control Problem for Systems Represented by Fractional Calculus

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Abstract

The systems are represented by fractional calculus in many problems of physical science; heat diffusion, neutron migration, RC network with infinte RC distributed lines, to mention only a few.

This paper discusses the optimal control of multivariable systems represented by fractional calclus. The optimal control problem of systems mentioned above has not been investigated sufficiently because of the necessity of the complicated mathematical operation.

To overcome this difficulty, I suggest to define Generalized Exponential Function which is easily extended to matrix function. Using this function, systems are representable by input-output functional relations and as a result, the characteristic expansion technique can be applied to the optimal control problem in these systems.

1. Introduction

In many problems of physical science that we encounter, fractional calculus plays a paramount role, i.e.; heat conduction[2] (Carslaw and Jaeger, 1947), diffusion[3] (Crank, 1956), viscous flow[4] (Moore, 1964), neutron migration[5] (Davison, 1957), flow through porous media[6] (Muskat, 1937), the simplification of the high order differential equations, the complex control systems[11] (Manabe, 1960) and the realization of fractional integral computing elements[12] (Hashimoto, 1969), to mention only a few.

The purpose of this paper is to present optimal control problems of the systems represented by fractional calculus. Optimal control problems have been studied by considering the system dynamics as described by a set of differential equations. Many physical systems, however, are often identified by means of input-output functional relations. We discuss a kind of a direct method for solving a class of minimum energy control problems applicable to systems of fractional calculus representable by input-output functional relations.

The paper is organized as follows. In Section II, to represent the system by input-output functional relations, Generalized Exponential Function and Generalized Matrix Exponential Function are defined. These properties are also discussed in this section. In Section III, the class of systems and the type of optimal control problems under consideration are presented. In Section IV, the concept of the characteristic set of given function is defined and an expansion technique in terms of the characteristic set is outlined. In Section V, the expansion technique is applied to an optimal control problem of systems represented by fractional calculus. In Section VI, a simple illustrative example is discussed.

Received September 30 1977

2. Generalized Exponential Function and Generalized Matrix Exponential Function

2–1 Definition of Generalized nth degree Exponential Function

We define Generalized nth degree Exponential Function.[10)]

$$\mathcal{E}^n(a, \nu, t) = \mathcal{E}_r^n(r, \theta, \nu, t) + j\,\mathcal{E}_i^n(r, \theta, \nu, t) \tag{1}$$

$$\left.\begin{aligned} \mathcal{E}_r^n(r, \theta, \nu, t) &= \sum_{m=0}^{\infty} \frac{(m+n-1)!\, r^m \cos m\theta}{m!\,(n-1)!\,\Gamma[(m+n)\nu]} t^{(m+n)\nu-1} \\ \mathcal{E}_i^n(r, \theta, \nu, t) &= \sum_{m=0}^{\infty} \frac{(m+n-1)!\, r^m \sin m\theta}{m!\,(n-1)!\,\Gamma[(m+n)\nu]} t^{(m+n)\nu-1} \end{aligned}\right\} \tag{2}$$

where $a = re^{j\theta}$ in polar form, ν is fractional calculus constant and $\mathcal{E}_r^n(r, \theta, \nu, t)$ and $\mathcal{E}_i^n(r, \theta, \nu, t)$ are respectively real part and imaginary part of $\mathcal{E}^n(a, \nu, t)$ respectively.

2–2 Basic properties of Generalized nth degree Exponential Function

Basic properties of Generalized nth degree Exponential Function are listed in Table 1.

Table 1 Basic properties of Generalized nth degree Exponential Function

No.	Basic Properties	
1	$\mathcal{E}^n(r, \theta, \nu, t) = \frac{1}{\lvert r\rvert^{n-1/\nu}} \mathcal{E}^n(1, \theta, \nu, \lvert r\rvert^{1/\nu} t)$	$n \geq 2$
2	$\int_0^t \mathcal{E}^n(a, \nu, t)(dt)^\nu = \frac{1}{a}\left\{\mathcal{E}^n(a, \nu, t) - \int_0^t \mathcal{E}^{n-1}(a, \nu, t)(dt)^\nu\right\}$	$n \geq 2$
3	$\int_0^t \mathcal{E}^n(a, \nu, t-\tau)\, u(\tau)\, d\tau = \frac{1}{a}\left\{\sum_{m=0}^{\infty} \frac{(m+n-1)!\, a^m\, t^{(m+n-1)\nu}}{m!\,(n-1)!\,\Gamma[(m+n-1)\nu+1]} - \int_0^t \mathcal{E}^{n-1}(a, \nu, t-\tau) u(\tau) d\tau\right\}$	$n \geq 1$ $u(t)$: step fnc.

(The proof is derived in Appendex A)

2–3 Definition of Generalized Matrix Exponential Function

We can define Generalized Matrix Exponential Function for the complex square matrix A formally substituted for a in Generalized Exponential Function, $\mathcal{E}(a, \nu, t)$

$$\mathcal{E}^n(\boldsymbol{A}, \nu, t) = \sum_{m=0}^{\infty} \frac{(m+n-1)!\, \boldsymbol{A}^m}{m!\,(n-1)!\,\Gamma[(m+n)\nu]} t^{(m+n)\nu-1} . \tag{3}$$

2–4 Basic properties of Generalized Matrix Exponential Function

(i) The following property holds if $\boldsymbol{P}$ is any real nonsingular constant matrix.

$$\boldsymbol{P}^{-1}\mathcal{E}^n(\boldsymbol{A}, \nu, t)\,\boldsymbol{P} = \mathcal{E}^n(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}, \nu, t) \tag{4}$$

and $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ can be transformed into the Jordan canonical form $\boldsymbol{J}$ for an arbitrary $\boldsymbol{P}$.

(The proof is derived in Appendex B.)

(ii) If $\boldsymbol{J}$ is the canonical form of matrix **A**, which yields

$$\boldsymbol{J} = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_k \end{bmatrix} = \mathrm{diag}\,[\lambda_1 \cdots \lambda_k]$$

where i=1, ...k, and λ_i, are distinct eigenvalues.
Then, we obtain

$$\mathcal{E}^n(\boldsymbol{J}, \nu, t) = \begin{bmatrix} \mathcal{E}^n(\lambda_1, \nu, t) & & & \boldsymbol{0} \\ & \mathcal{E}^n(\lambda_2, \nu, t) & & \\ & & \ddots & \\ \boldsymbol{0} & & & \mathcal{E}^n(\lambda_\kappa, \nu, t) \end{bmatrix} . \tag{5}$$

(The proof is derived in Appendex C.)
(iii) In the case of multiple eigenvalues the system matrix of order (k×k) is diagonalized in a type of Jordan form as

$$\boldsymbol{J} = \begin{bmatrix} \lambda & 1 & 0 & & \\ & \lambda & 1 & 0 & \\ & & \ddots & & \ddots \\ & \boldsymbol{0} & & \ddots & \\ & & & & \lambda \end{bmatrix} .$$

If the order of the multiplicity is k, the Jordan matrix will be of dimension (k×k) and by substituting **J** into eqn. (3), we obtain

$$\mathcal{E}^n(\boldsymbol{J}, \nu, t) = \begin{bmatrix} \mathcal{E}^n(\lambda, \nu, t), & \mathcal{E}^{n+1}(\lambda, \nu, t), & \cdots\cdots, & \mathcal{E}^{n-\kappa-1}(\lambda, \nu, t) \\ & \ddots & & \\ \boldsymbol{0} & & \ddots & \mathcal{E}^{n+1}(\lambda, \nu, t) \\ & & & \mathcal{E}^n(\lambda, \nu, t) \end{bmatrix} . \tag{6}$$

(The proof is given in Appendex D.)

3. The Problem

3–1 The system represented by fractional calculus

The system represented by fractional calculus is expressed by the following matrix differential equation.

$$\frac{d^\nu}{dt^\nu} \boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \tag{7}$$

where $\boldsymbol{x}$ represents quasi-state variables of n dimension vector,
$\boldsymbol{u}$ represents input of r dimension vector,
$\boldsymbol{y}$ represents output of m dimension vector,
and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are the constant matrix of dimension (n×n), (n×r), (m×m) respectively.

In systems of fractional calculus, quasi-state variables do not supply sufficient information about the system viewed in conventional sense. Because it is clear that these systems are closely related to distributed parameter systems with infinite dimensions.

To solve these equations we assume for simplicity reasons that all the eigenvalues are distinct and real and also the system is at rest. We state that

$$\boldsymbol{\xi}(t) = \boldsymbol{P}\boldsymbol{x}(t) \tag{8}$$

where $\boldsymbol{P}$ is a real nonsingular constant matrix.

From (8), we substitute new variables in place of $\boldsymbol{x}$ in (7) to obtain

$$\frac{d^\nu}{dt^\nu} \boldsymbol{\xi}(t) = \boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}^{-1} \boldsymbol{\xi}(t) + \boldsymbol{P}\boldsymbol{B}\boldsymbol{u}(t) \tag{9}$$

P matrix is taken such that to give

$$\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}=\boldsymbol{J}=\mathrm{diag}\,[\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_\kappa].$$

In this case all the eigenvalues are taken as distinct. When there are multiple eigenvalues, then **J** takes Jordan form.

Applying the Laplace transformation to (9) and taking into account that the system is at rest, we obtain

$$(s^\nu \boldsymbol{I}-\boldsymbol{J})\,L\,\{\boldsymbol{\xi}\,(t)\}=\boldsymbol{P}\boldsymbol{B}\,L\,\{\boldsymbol{u}\,(t)\}. \tag{10}$$

By pre-multiplying bothsides with $(s^\nu\boldsymbol{I}-\boldsymbol{J})^{-1}$, we obtain

$$L\,\{\boldsymbol{\xi}\,(t)\}=(s^\nu \boldsymbol{I}-\boldsymbol{J})^{-1}\boldsymbol{P}\boldsymbol{B}\,L\,\{\boldsymbol{u}\,(t)\} \tag{11}$$

In (11) the factor $(s^\nu\boldsymbol{I}-\boldsymbol{J})^{-1}$ is the fractional characteristic transfer matrix, which is

$$(s^\nu \boldsymbol{I}-\boldsymbol{J})^{-1}=\begin{pmatrix}\dfrac{1}{s^\nu-\lambda_1} & & \boldsymbol{0}\\ & \ddots & \\ \boldsymbol{0} & & \dfrac{1}{s^\nu-\lambda_\kappa}\end{pmatrix} \tag{12}$$

Inverse Laplace transform of (12) is

$$L^{-1}\,[(s^\nu \boldsymbol{I}-\boldsymbol{J})^{-1}]=\begin{pmatrix}\mathcal{E}\,(\lambda_1,\ \nu,\ t) & & \boldsymbol{0}\\ & \ddots & \\ \boldsymbol{0} & & \mathcal{E}\,(\lambda_\kappa,\ \nu,\ t)\end{pmatrix}=\mathcal{E}\,(\boldsymbol{J},\ \nu,\ t) \tag{13}$$

On taking inverse Laplace transform of eqn. (11) and from the relation given in (8) we obtain

$$\boldsymbol{x}\,(t)=\int_0^t \mathcal{E}\,(\boldsymbol{A},\ \nu,\ t-\tau)\,\boldsymbol{B}\boldsymbol{u}\,(\tau)\,d\,\tau$$

This equation is derived without considering the initial value, but if we consider all the past $(-\infty,\ 0)$ of inputs, we obtain

$$\begin{aligned}\boldsymbol{x}\,(t)&=\int_{-\infty}^0 \mathcal{E}\,(\boldsymbol{A},\ \nu,\ t-\tau)\,\boldsymbol{B}\boldsymbol{u}\,(\tau)\,d\,\tau\\&\quad+\int_0^t \mathcal{E}\,(\boldsymbol{A},\ \nu,\ t-\tau)\,\boldsymbol{B}\boldsymbol{u}\,(\tau)\,d\tau\end{aligned} \tag{14}$$

and

$$\begin{aligned}\boldsymbol{y}\,(t)&=\boldsymbol{C}\int_{-\infty}^0 \mathcal{E}\,(\boldsymbol{A},\ \nu,\ t-\tau)\,\boldsymbol{B}\boldsymbol{u}\,(\tau)\,d\,\tau\\&\quad+\boldsymbol{C}\int_0^t \mathcal{E}\,(\boldsymbol{A},\ \nu,\ t-\tau)\,\boldsymbol{B}\boldsymbol{u}\,(\tau)\,d\,\tau\end{aligned} \tag{15}$$

3-2 The problem

The class of systems considered in this paper is the class of realizable multi-input-multi-output continuous linear systems, which are representable by input-output functional relation of the form

$$y(t) = h_0(t) + \int_{t_0}^{t} H(t;\tau)\, u(\tau)\, d\tau \tag{16}$$

where the output $\mathbf{y}(t)$ and the input $\mathbf{u}(t)$ are q and r vectors, respectively, $\mathbf{H}(t, \tau)$ is the q× r matrix function, (kj)th entry is $\mathbf{h}_j^k(t;\tau)$ and this function corresponds to $\mathbf{C}\,\mathcal{E}(\mathbf{A}, \nu, t-\tau)\,\mathbf{B}$ in (15). The initial and terminal conditions are

$$y(t_0) = y_0 = h_0(t_0) \in R_q \tag{17}$$

and

$$y(T) = y_T = h_0(T) + \int_{t_0}^{T} H(T;\tau)\, u(\tau)\, d\tau \in R_q$$

respectively. Let R_q be the Euclidean q space, and let $z_T = y_T - h_0(T)$, and then the relation between z_T and u is linear.

Let $L_{2,r}$ be the space of r-vector valued functions, each component of a member vector being square integrable on the interval $[t_0, T]$. Let an r×r matrix $Q(t)$ be defined for $t \in [t_0, T]$ such that it is symmetric and positive definite for each t, and that $q_k \in L_{2,r}$ for k=1, 2,...., r, where q_k is the kth column of $Q(t)$. Let the transpose of a vector or a matrix be denoted by a prime.

The type of optimal control problem investigated in this paper is as follows; given a linear system described by (16) and an arbitrary initial output $y(t_0) = y_0 \in R_q$, find the input $u(t)$, $t_0 \leq t \leq T$ and $u \in L_{2,r}$, such that 1) the terminal output $y(T) = y_T \in R_q$, and 2) the cost functional given by

$$J = \int_{t_0}^{T} u'(t)\, Q(t)\, u(t)\, dt \tag{18}$$

is minimized. The terminal time T is assumed fixed. The terminal output y_T may be fixed or given in terms of T. In what follows, the matrix $Q(t)$ in (18) will be taken to be I, the identity matrix. This does not constitute any loss of generality. For a positive definite symmetric matrix $Q(t)$, there exists a nonsingular matrix $P(t)$ such that

$$Q(t) = P'(t)\, P(t) \tag{19}$$

Thus, if

$$v(t) = P(t)\, u(t) \tag{20}$$

and

$$K(t;\tau) = H(t;\tau)\, P^{-1}(\tau) \tag{21}$$

then the linear system given by

$$y(t) - h_0(t) = \int_{t_0}^{t} H(t;\tau)\, u(\tau)\, d\tau \tag{22}$$

and the cost functional given by (18) may be transformed to, respectively,

$$y(t) - h_0(t) = \int_{t_0}^{t} K(t;\tau)\, v(\tau)\, d\tau \tag{23}$$

and

$$J = \int_{0}^{T} v'(t)\, v(t)\, dt \tag{24}$$

The optimal input for the transformed problem $\boldsymbol{v}^*(t)$ is found by the characteristic expansion method to be developed later and the actual optimal input $\boldsymbol{u}^*(t)$ is uniquely determined by

$$\boldsymbol{u}^*(t) = \boldsymbol{P}^{-1}(t)\,\boldsymbol{v}^*(t) \tag{25}$$

The optimal control problem (P) may then be stated as: given an element $\boldsymbol{z}_T \in \boldsymbol{R}_q$ and a bounded linear operator L from $\boldsymbol{L}_{2,r}$ into $\boldsymbol{R}_q$ defined by

$$L\boldsymbol{u} = \int_{t_0}^{T} \boldsymbol{H}(T;t)\,\boldsymbol{u}(t)\,dt, \tag{26}$$

find an element $\boldsymbol{u} \in \boldsymbol{L}_{2,r}$ that minimizes

$$\|\boldsymbol{u}\|_{I2,r} = \int_{t_0}^{T} \boldsymbol{u}'(t)\,\boldsymbol{u}(t)\,dt \tag{27}$$

subject to

$$L\boldsymbol{u} = \boldsymbol{z}_T \tag{28}$$

The solution of this problem can be obtained using the notion of the characteristic set introduced in Section IV.

4 Characteristic Set of Given Functions-Definitions

Let $\boldsymbol{L}_2$ denote the Hilbert space of square integrable functions on $[t_0, T]$. Suppose that a finite set of functions, $\{F_i(t)\,|\,F_i \in \boldsymbol{L}_2;\ i=1, 2, \cdots, p,\}$ is given. If there is a finite set of linearly independent functions $\{f_m(t)\,|\,f_m \in \boldsymbol{L}_2;\ m=1, 2, \cdots, s\}$ such that each member of the set $\{F_i(t)\}$ is expressible as a finite linear combination of the functions of this set, i.e.,

$$F_i(t) = \sum_{m=1}^{(m)i} a_m{}^i f_m(t) \qquad i = 1, 2, \cdots, p \tag{29}$$

then the finite set $\{f_m(t)\}$ is called a characteristic set of the set of functions $\{F_i(t)\}$.

Assertion 1

Given an arbitrary finite set of functions $\{F_i(t)\,|\,F_i \in \boldsymbol{L}_2;\ i=1, 2, \cdots, p\}$, there exists at least one characteristic set.

Consider the set of functions consisting of all the entries in the jth column of $\boldsymbol{H}(T, t)$. By assumption, each element of this set is in $\boldsymbol{L}_2$. From Assertion 1, there exists a characteristic set. Let the complete orthonormal set in $\boldsymbol{L}_2$ constructed from the characteristics set be denoted by $\{\phi_{jm}(t)\}$. Repeating for each j, j=1, 2,...., r, the sets $\{\phi_{1m}(t)\}$, $\{\phi_{2m}(t)\}$ and $\{\phi_{jm}(t)\}$ are obtained. In this paper the complete orthonormal set can be constructed by Schmidt's orthogonalization.

5 The characteristic expansion technique

Lemma 1

Given (26) and (28), there exist orthonormal sets of functions $\{\phi_{jm}(t)\}$, j=1, 2,...., r, each complete in $\boldsymbol{L}_2$; a set of integers (m1), (m2), ... and (mr); and constants $A_{jm}{}^k<\infty$, for k=1, 2, ..., q, j=1, 2, ..., r, and m=1, 2,...., (mj); such that, for k=1, 2,...., q.

$$z_T^{\kappa} = \sum_{j=1}^{r} \sum_{m=1}^{(mj)} A_{jm}^{\kappa} \int_{t_0}^{T} \phi_{jm}(t)\, u_j(t)\, dt$$

(The proof is abbreviated here.)[7]
Define the matrices $\boldsymbol{A}_j=[A_{jm}^k]$, for $j=1, 2, \cdots, r$ and define $q\times n$ matrix $\boldsymbol{A}$ by

$$\boldsymbol{A} = [\boldsymbol{A}_1 \vdots \boldsymbol{A}_2 \vdots \cdots\cdots \vdots \boldsymbol{A}_r] \tag{30}$$

where

$$n = \sum_{i=1}^{r} (m_j) \tag{31}$$

Since $\{\phi_{jm}(t)\}$ is complete in $\boldsymbol{L}_2$, an arbitrary element $u_j \in \boldsymbol{L}_2$ can be uniquely expressed by

$$\boldsymbol{u}_j(t) = \sum_{m=1}^{\infty} A_{jm}\, \phi_{jm}(t). \tag{32}$$

Repeating, for j=1, 2,...., r, for an arbitrary element $\boldsymbol{u} \in \boldsymbol{L}_{2,r}$,

$$\boldsymbol{u}'(t) = \left[\sum_{m=1}^{\infty} B_{1m}\, \phi_{1m}(x),\ \sum_{m=1}^{\infty} B_{2m}\, \phi_{2m}(t),\ \cdots\cdots,\ \sum_{m=1}^{\infty} B_{rm}\, \phi_{rm}(t)\right] \tag{33}$$

i.e., a $\boldsymbol{u} \in \boldsymbol{L}_{2,r}$ uniquely defines a set of constants B_{jm} and hence an n vector,

$$\boldsymbol{b}' = [B_{11}, B_{12}, \cdots, B_{1(m1)} \cdots B_{r1}, B_{r2}, \cdots, B_{r(mr)}] \tag{34}$$

Conversely, an n vector $\boldsymbol{b}$ and constants B_{jm} for j=1, 2,...., r, and m=(mj)+1, (mj)+2, ... such that

$$\sum_{i=1}^{n} b_i^2 + \sum_{j=1}^{r} \sum_{m=(mj)+1}^{\infty} B_{jm}^2 < \infty \tag{35}$$

uniquely define an element in $\boldsymbol{L}_{2,r}$.

Using (34), it is seen that (20) is reduced to

$$z_T = \boldsymbol{A}\boldsymbol{b}. \tag{36}$$

Any element in $\boldsymbol{L}_{2,r}$ that satisfies (36) is a feasible input. Concerning the existence of a feasible input, the following lemma holds.

Lemma 2

Given the problem (P), there exists a feasible input in $\boldsymbol{L}_{2,r}$ if rank $\boldsymbol{A}$=q.
(The proof is abbreviated here.)
From (33) and (34), it is seen that the cost functional may be written as

$$\boldsymbol{J} = \boldsymbol{b}'\boldsymbol{b} + \sum_{j=1}^{r} \sum_{m=(mj)+1}^{\infty} B_{jm}^2 \tag{37}$$

The following theorems contain the solution of the problem.

Theorem 1

Given the problem (P), if rank $\boldsymbol{A}$=q, then there exists an optimal input $\boldsymbol{u}^* \in \boldsymbol{L}_{2,r}$ and it is unique.

Theorem 2

Given the problem (P), let $\boldsymbol{A}$ be a q×q nonsigular submatrix of $\boldsymbol{A}$ containing the first q columns of $\boldsymbol{A}$. Let $z_T=\bar{\boldsymbol{A}}\bar{\boldsymbol{b}}+\hat{\boldsymbol{A}}\hat{\boldsymbol{b}}$. Then the optimal input is given by

$$\boldsymbol{b}^* = \begin{bmatrix} \boldsymbol{C} - \boldsymbol{D}(\boldsymbol{D}'\boldsymbol{D} + \boldsymbol{I})^{-1}\boldsymbol{D}'\boldsymbol{C} \\ (\boldsymbol{D}'\boldsymbol{D} + \boldsymbol{I})^{-1}\boldsymbol{D}'\boldsymbol{C} \end{bmatrix} \tag{38}$$

$$B^*_{jm} = 0 \text{ for } j = 1, 2, \cdots, r, \ m = (mj) + 1, \ (mj) + 2, \cdots$$

where

$$\boldsymbol{C} = \bar{\boldsymbol{A}}^{-1} \boldsymbol{z}_T, \ \boldsymbol{D} = \bar{\boldsymbol{A}}^{-1} \hat{\boldsymbol{A}}.$$

(The proofs of Theorem 1 and Theorem 2 are abbreviated here.[8])

The procedure developed in the above is defined as the characteristic expansion method.

6 The simple illustrative example

Consider the optimal control problem of the system expressed by the following fractional differential equation,

$$\frac{d^{1.5}}{dt^{1.5}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{39}$$

Now suppose the initial state is neglected, the following input-output relation can be established from (39).

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \int_0^t \begin{bmatrix} \mathcal{E}(-1, 1.5, t-\tau) \\ \mathcal{E}(-2, 1.5, t-\tau) \end{bmatrix} u(\tau)\, d\tau \tag{40}$$

The problem:

We find the optimal input such that 1) the terminal output $y_1(1)=1$, $y_2(1)=1$, and 2) the cost functional given by

$$\boldsymbol{J} = \int_0^1 u^2(\tau)\, d\tau \tag{41}$$

The procedure:

The impulse response of this system is expressed by the Generalized Exponential Function as follows.

$$\boldsymbol{H}(t, \tau) = \begin{bmatrix} \mathcal{E}(-1, 1.5, t-\tau) \\ \mathcal{E}(-2, 1.5, t-\tau) \end{bmatrix} \tag{42}$$

Secondly, the following complete orthonormal set can be obtained by Schmidt's orthogonalization.

$$\phi_1(\tau) = \frac{\mathcal{E}(-1, 1.5, 1-\tau)}{\sqrt{\int_0^1 \{\mathcal{E}(-1, 1.5, 1-\tau)\}^2 d\tau}}$$

$$\phi_2(\tau) = \frac{\mathcal{E}(-1, 1.5, 1-\tau) - \left\{\int_0^1 \mathcal{E}(-2, 1.5, 1-\tau)\, \phi_1(\tau)\, d\tau\right\} \phi_1(\tau)}{\sqrt{\int_0^1 \left[\mathcal{E}(-2, 1.5, 1-\tau) - \left\{\int_0^1 \mathcal{E}(-2, 1.5, 1-\tau)\, \phi_1(\tau)\, d\tau\right\} \phi_1(\tau)\right]^2 d\tau}} \tag{43}$$

We obtain the coefficients of this complete orthonormal set.

$$\begin{aligned} a_{11} &= \int_0^1 \mathcal{E}(-1,\ 1.5,\ 1-\tau)\ \phi_1(\tau)\ d\tau = 0.705 \\ a_{12} &= \int_0^1 \mathcal{E}(-1,\ 1.5,\ 1-\tau)\ \phi_2(\tau)\ d\tau = 0 \\ a_{21} &= \int_0^1 \mathcal{E}(-2,\ 1.5,\ 1-\tau)\ \phi_1(\tau)\ d\tau = 0.588 \\ a_{22} &= \int_0^1 \mathcal{E}(-2,\ 1.5,\ 1-\tau)\ \phi_2(\tau)\ d\tau = 0.093 \end{aligned} \tag{44}$$

From (43) and (44), the rank of matrix $A=\begin{bmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{bmatrix}$ is 2.

Therefore, Theorem 1 is established.
The relation between $\boldsymbol{z}_T$ and $\boldsymbol{b}$ is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.705 & 0 \\ 0.588 & 0.093 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{45}$$

From (45), we obtain

$$B_1 = 1.4165,\ B_2 = 1.7827$$

Using Theorem 2, we can obtain the optimal input $\boldsymbol{u}^*(\tau)$ and the minimum cost functional $\boldsymbol{J}$ as follows.

$$\begin{aligned} \boldsymbol{u}^*(\tau) &= 1.4165\,\phi_1(\tau) + 1.7827\,\phi_2(\tau) \\ &= -13.88\,\mathcal{E}(-1,\ 1.5,\ 1-\tau) + 19.07\,\mathcal{E}(-2,\ 1.5,\ 1-\tau) \end{aligned} \tag{46}$$

$$\boldsymbol{J} = \int_0^1 u^2(\tau)\ d\tau = B_1^2 + B_2^2 = 5.1848 \tag{47}$$

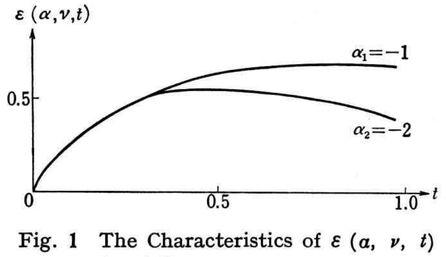

Fig. 1 The Characteristics of $\mathcal{E}$ (a, ν, t) (ν=1.5)

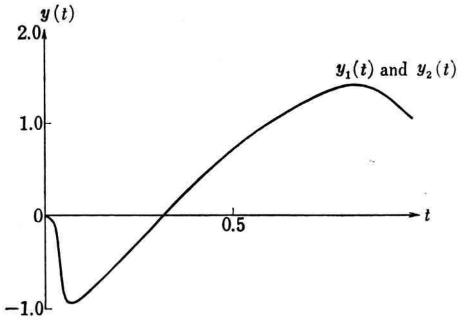

Fig. 2 The Outputs $y_1(t)$ and $y_2(t)$

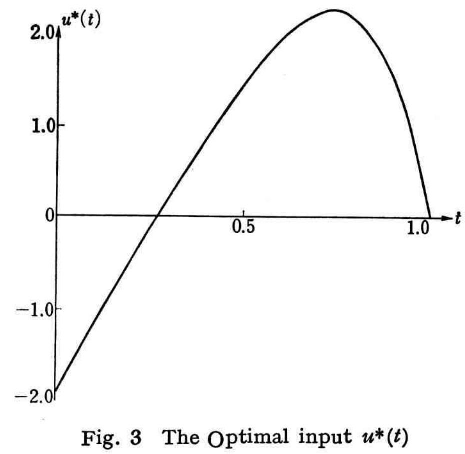

Fig. 3 The Optimal input $u^*(t)$

Fig. 1 illustrates the characteristics of $\mathcal{E}(-1, 1.5, t)$ and $\mathcal{E}(-2, 1.5, t)$.
Fig. 2 illustrates the characteristics of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$.
Fig. 3 illustrates the characteristics of the optimal input $\boldsymbol{u}^*$.

7 Conclusions

I have presented the method to solve the optimal control problem represented by fractional calculus without using the conventional method. This method is based on defining the new function for systems of fractional calculus and using the characteristic expansion technique reducing the optimal control problem to that of solving a finite set of algebraic equations. But we have the problem in case of $\nu \leqq 0.5$ in (1). The reason is that $\mathcal{E}(a, \nu, t)$ is not square integrable for $\nu \leqq 0.5$.

To solve this problem, we need to multiply $\mathcal{E}(a, \nu, t)$ by an arbitrary weight function, then can make it possible to be square integrable. In this paper, though this discussion is abbreviated, the solution in this case can be easily obtained by a simple operation in the same manner as discussed in this paper.

Acknowledgement

The author thanks Dr. SHINICHI TAKAHASHI, associate professor of KEIO University for several comments.

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APPENDIX

(A-1) Derivation of the property No. 1 of Generalized nth degree Exponential Function

Now we consider the real part of this function.

$$\mathcal{E}_r{}^n(r, \theta, \nu, t) = \sum_{m=0}^{\infty} \frac{(m+n-1)!\, r^m \cos m\theta\, t^{(m+n)\nu-1}}{m!\,(n-1)!\,\Gamma[(m+n)\nu]}$$

$$= \frac{1}{|r|^{n-1/\nu}} \sum_{m=0}^{\infty} \frac{(m+n-1)!\cos m\theta}{m!\,(n-1)!\,\Gamma[(m+n)\nu]} (|r|^{1/\nu}t)^{(m+n)\nu-1} \tag{A1}$$

$$= \frac{1}{|r|^{n-1/\nu}} \mathcal{E}_r^{\,n}(1, \theta, \nu, |r|^{1/\nu}t)$$

In the same manner, the relation of the imginary part is derived.

(A–2) Derivation of the property No. 2

$$L\left[\int_0^t \mathcal{E}^n(a, \nu, t)\,(dt)^\nu\right] = \frac{1}{(s^\nu - a)^n}\frac{1}{s^\nu}$$

$$= \frac{1}{a}\left\{\frac{1}{(s^\nu - a)^n} - \frac{1}{(s^\nu - a)^{n-1}}\frac{1}{s^\nu}\right\} \tag{A2}$$

$$\int_0^t \mathcal{E}^n(a, \nu, t)\,(dt)^\nu = \frac{1}{a}\left\{\mathcal{E}^n(a, \nu, t) - \int_0^t \mathcal{E}^{n-1}(a, \nu, t)\,(dt)^\nu\right\}$$

(A–3) Derivation of the property No. 3

$$L\left[\int_0^t \mathcal{E}^n(a, \nu, t-\tau)\,u(\tau)\,d\tau\right] = \frac{1}{(s^\nu - a)^n}\cdot\frac{1}{s}$$

$$= \frac{1}{a}\left\{\frac{s^{\nu-1}}{(s^\nu - a)^n} - \frac{1}{(s^\nu - a)^{n-1}}\frac{1}{s}\right\}$$

$$\int_0^t \mathcal{E}^n(a, \nu, t-\tau)\,u(\tau)\,d\tau = \frac{1}{a}\left\{\sum_{m=0}^{\infty}\frac{(m+n-1)!\,a^m\,t^{(m+n-1)\nu}}{m!\,(n-1)!\,\Gamma[(m+n-1)\nu+1]} - \int_0^t \mathcal{E}^{n-1}(a, \nu, t-\tau)\,u(\tau)\,d\tau\right\} \tag{A3}$$

(B) Derivation of eqn. (4)

From eqn. 3,

$$\mathcal{E}^n(A, \nu, t) = \frac{t^{n\nu-1}}{\Gamma[n\nu]} I + \frac{n\,t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]} A + \frac{(n+1)!\,t^{(n+2)\nu-1}}{2\,(n-1)!\,\Gamma[(n+2)\nu]} A^2 + \cdots\cdot \tag{B1}$$

From (B1),

$$P^{-1}\mathcal{E}^n(A, \nu, t)\,P = P^{-1}\frac{t^{n\nu-1}}{\Gamma[n\nu]} I P + P^{-1}\frac{n\,t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]} A P + P^{-1}\frac{(n+1)!\,t^{(n+2)\nu-1}}{2\,(n-1)!\,\Gamma[(n+2)\nu]} A^2 P + \cdots\cdot$$

$$= \frac{t^{n\nu-1}}{\Gamma[n\nu]} I + \frac{n\,t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]} P^{-1} A P + \frac{(n+1)!\,t^{(n+2)\nu-1}}{2\,(n-1)!\,\Gamma[(n+2)\nu]} (P^{-1} A P)^2 + \cdots\cdots \tag{B2}$$

$$= \mathcal{E}^n(P^{-1} A P, \nu, t)$$

where $\quad P^{-1} A^2 P = P^{-1} A \cdot A P = P^{-1} A P P^{-1} A P = (P^{-1} A P)^2$

(C) Derivation of eqn. (5)

From eqn. (3)

$$\mathcal{E}^n(J, a, t) = \begin{bmatrix} 1 & & O \\ & 1 & \\ O & \ddots & \\ & & 1 \end{bmatrix} \frac{t^{n\nu-1}}{\Gamma(n\nu)} + \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2. & \\ O & \ddots & \\ & & \lambda_k \end{bmatrix} \frac{n\, t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]} + \cdots$$

$$+ \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ & \ddots & \\ O & & \lambda_k \end{bmatrix}^2 \frac{(n+1)!\, t^{(n+2)\nu-1}}{2(n-1)!\,\Gamma[(n+2)\nu]} + \cdots.$$

$$= \begin{bmatrix} 1 & & O \\ & 1 & \\ O & & \\ & & 1 \end{bmatrix} \frac{t^{n\nu-1}}{\Gamma(n\nu)} + \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & \ddots & \\ & & \lambda_k \end{bmatrix} \frac{n\, t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]} + \cdots\cdots$$

$$+ \begin{bmatrix} \lambda_1^2 & & O \\ & \lambda_2^2 & \\ O & \ddots & \\ & & \lambda_k^2 \end{bmatrix} \frac{(n+1)!\, t^{(n+2)\nu-1}}{2(n-1)!\,\Gamma[(n+2)\nu]} + \cdots\cdots$$

$$\begin{pmatrix} \left(\frac{t^{n\nu}}{\Gamma[n\nu]} + \frac{n\, t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]}\lambda_1 + \frac{(n+1)!\, t^{(n+2)\nu-1}}{2(n-1)!\,\Gamma[(n+2)\nu]}\lambda_1^2 + \cdots\right) & & O \\ & \left(\frac{t^{n\nu-1}}{\Gamma[n\nu]} + \frac{n\, t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]}\lambda_2 + \frac{(n+1)!\, t^{(n+2)\nu-1}}{2(n-1)!\,\Gamma[(n+2)\nu]}\lambda_2^2 + \cdots\right) & \\ O & \left(\frac{t^{n\nu-1}}{\Gamma[n\nu]} + \frac{n\, t^{(n+1)\nu-1}}{\Gamma[(n+1)\nu]}\lambda_k + \frac{(n+1)!\, t^{(n+2)\nu-1}}{2(n-1)!\,\Gamma[(n+2)\nu]}\lambda^2_k + \cdots\right) & \end{pmatrix} \tag{C1}$$

From (C1), we obtain the following result.

$$\mathcal{E}^n(J, \nu, t) = \begin{pmatrix} \mathcal{E}_2^n(\lambda_1, \nu, t) & \cdot & \cdot & \cdot & \cdot & \cdot & O \\ \cdot & \mathcal{E}_2^n(\lambda_2, \nu, t) & & & & & \cdot \\ \cdot & & \ddots & & & & \cdot \\ O & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{E}_2^n(\lambda_k, \nu, t) \end{pmatrix} \tag{C2}$$

(D) Derivation of eqn. (6)

$$\mathcal{E}_2^n(J, \nu, t) = \sum_{m=0}^{\infty} \frac{(m+n-1)!\, t^{(m+n)\nu-1}}{m!\,(n-1)!\,\Gamma[(m+n)\nu]} \begin{pmatrix} \lambda^m, m\lambda^{m-1}, \cdots, \frac{m!\lambda^{m-r}}{(m-r)!r!}, & \frac{m!}{(m-k+1)!} & \frac{\lambda^{m-k+1}}{(k-1)!} \\ \cdot \quad \cdot & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ & , \lambda^m, & m\lambda^{m-1} \\ O \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad , & \lambda^m \end{pmatrix} \tag{D1}$$

The (r+1)th element of the first row is

$$\sum_{m=0}^{\infty} \frac{(m+n-1)!\, t^{(m+n)\nu-1}}{m!\,(n-1)!\,\Gamma[(m+n)\nu]} \cdot \frac{m!}{(m-r)!\, r!} \lambda^{m-r}$$

$$= \sum_{m=0}^{\infty} \frac{(m+n+r-1)!\,\lambda^m}{m!\,(n+r-1)!\,\Gamma[(m+n+r)\nu]} t^{(m+n+r)\nu-1} \tag{D2}$$

$$= \mathcal{E}^{n+r}(\lambda, \nu, t)$$

where $m-r \geq 0$.

For every element, the relation of (D2) is established.