

Vibration-Rotation Interaction in Engineering Problems. II.

On an Application of the Normal Coordinate Method of the Simple System

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Abstract

One of the most important and insolvable problem in our knowledges is that of the vibration of the mechanical system. Usually, a creation of vibrations is accompanied by the rotation of a mechanical system. In practice, the extremely small vibrations of one part of the mechanical system rotated about one axis which is undergone unbalanced force are sometimes propagated all around the whole system with an amplified amplitude, and then the system is caused a serious damage by this vibration. The fundamental informations on the mechanical vibrations should be accumulated in our knowledges.

It is worthwhile attempt that the usual analysis by cartesian coordinate method is transformed into that of normal coordinate method by the linear algebra. This work is done along this line preliminarily. In here, the frequencies of vibrations labelled as ν or ω usually are expressed by $\lambda^{1/2}$ which is directly related to the restoring forces of the potential energies of vibration.

Introduction

In the course of time, most valuable knowledges on the vibrations, rotations and their interactions of the mechanical systems are given in literatures^{1,2)}. Much of remarkable progresses on these problems has been made in the last decade³⁾. If a lot of the fundamental sources of vibration of each moving system are analized using the modest type of measuring tools and electronic devices, most of these vibrations to be avoided are dissipated from the system by making use of the recent informations of the mechanical designs. However, experiences would tell us a creation of several of other sources of vibration by a more severe mechanical circumstances. It is of seriously important to solve the dissipations of an accursed vibration accompaning the mechanical motion.

On the vibrations, rotations and their interactions in molecules, enormous efforts have been paid by many workers in the world for analizing the spectroscopic and thermal behaviours of them. Using the quantum theory, must of knowledges are summarized in a famous book by Herzberg⁴⁾. An exceedingly valuable basis of the theoretical treatments are given on the molecular vibrations by E.B. Wilson⁵⁾, et. al., on the rotational spectra of molecules by Towns⁶⁾ and by Allen⁷⁾ on that of vibration-rotation interactions. Unfortunately, it has not yet been remained to be solved the forces connecting the atoms in molecule. The force constant so-called Hooke's constant in a harmonic potential field macroscopically is the most important property of the molecules as well as in matters.

One of the purposes of the present paper is to discuss the forces of the relations between

the factors determining the modes of vibration of a mechanically simple system and that of molecules. As far as possible, to elucidate the relations between the displacement coordinate and normal coordinate a linear two body vibrator is discussed here.

General Remarks

When the work integral depends merely on the initial and final positions of the system and not at all on the path acrossed over between the positions, the force is said to be conservative. Thus a conservative field of force may be characterized by the fact that the work done by it on a system which traverses any arbitrary closed path is zero. In this system, the applied forces are essentially determined by the potential energy V which is the function of the coordinate (q_1, q_2, \dots, q_N) . When an applied force on the system is derivable from a potential energy V , the vanishing of the virtual force for all virtual displacements means the relations

$$\frac{\partial V}{\partial q_i} = 0 \quad i = 1, 2, \dots, N \quad (1)$$

q 's being the independent coordinats of the system. For this reason, V has a stationary value at a position of equilibrium.

Then we suppose the infinitesimal displacements at near the equilibrium position. The potential energy will be some function of the displacements. It may be expressed as a power series in the displacement q_i .

$$\begin{aligned} 2V &= 2V_0 + 2 \sum_{i=1}^N \left(\frac{\partial V}{\partial q_i} \right) q_i + \sum_{i,j=1}^N \left(\frac{\partial^2 V}{\partial q_i \partial q_j} \right) q_i q_j + \text{higher terms} \\ &= 2V_0 + 2 \sum_{i=1}^N f_i q_i + \sum_{i,j=1}^N f_{ij} q_i q_j + \text{higher terms} \end{aligned} \quad (2)$$

By choosing that the equilibrium postion is a reference frame work, V_0 may be eliminated. Furthermore, the system is in their equilibrium postions, so that

$$\left(\frac{\partial V}{\partial q_i} \right) = f_i = 0 \quad i = 1, 2, \dots, N \quad (3)$$

For sufficiently small amplitudes of vibration, the higher terms may be neglected. Then we have

$$2V = \sum_{i,j=1}^N f_{ij} q_i q_j \quad (4)$$

in which the f_{ij} 's are constants given by

$$f_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j} \right) \quad (5)$$

and with $f_{ij} = f_{ji}$.

The kinetic energy is expressed as a quadratic function in the time derivatives of the displacement coordinates q_i , i.e.,

$$2T = \sum_{i,j=1}^N m_{ij} \dot{q}_i \dot{q}_j \quad (6)$$

Lagrange's equation of motion for the infinitesimal vibration about the equilibrium position may be written in the form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial V}{\partial q_i} = 0 \quad i = 1, 2, \dots, N \quad (7)$$

since T is only a function of the velocities in this coordinate system and V is that of the coordinates. Substituting of the expressions T and V given above yields the equations

$$\ddot{q}_j + \sum_{i=1}^N k_{ij} q_i = 0 \quad j = 1, 2, \dots, N \quad (8)$$

where $k_{ij} = f_{ij}/m_{ij}$.

This is a set of N simultaneous second order linear differential equations. One possible solution is

$$q_i = A_i \cos (\lambda^{1/2} t + \alpha) \quad (9)$$

where A_i , λ and α are properly chosen constants.

In general, the vibrational frequencies of a harmonic oscillator is given by

$$\nu_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (10)$$

in which k is called the spring constant and m the mass of the oscillator. The $\lambda^{1/2}$ in Eq. (9) is used to correspond to the quantity of the square root of (k/m) , and it is related to the frequency ν_0 as above Eq. (10).

From Eqs. (8) and (9),

$$\sum_{i=1}^N (k_{ij} - \delta_{ij} \lambda) A_i = 0 \quad j = 1, 2, \dots, N \quad (11)$$

in which δ_{ij} is the Kronecker delta. Eq. (11) is a set of simultaneous homogenous linear equation containing the N unknown amplitudes A_i . The special values of λ in Eq. (11) have non-zero solutions, and for all other values of λ 's, $A_i = 0$, $i = 1, 2, \dots, N$. The above equations result the determinantal or secular equation

$$\begin{vmatrix} k_{11} - \lambda & k_{12} & \dots & k_{1N} \\ k_{12} & k_{22} - \lambda & \dots & k_{2N} \\ \dots & \dots & \dots & \dots \\ k_{N1} & k_{N2} & \dots & k_{NN} - \lambda \end{vmatrix} = 0 \quad (12)$$

For the above determinant is composed of the coefficients of the unknown amplitudes A_i , in the set of equations (11), a fixed value of λ can be chosen so as to become the determinant to be zero. Then we are possible to obtain a solution, A_{ik} , for which the additional subscript k may be employed to express a corresponding particular value of λ_k . Such a set of equations does not determine the A_{ik} uniquely, but provides only their ratios. Therefore, when any set of l_{ik} may arbitrarily be selected to satisfy the mathematical solutions of the quantities designated by the coefficients of the secular equation by the relations,

$$l_{ik} = \frac{A_{ik}}{[\sum_i (A_{ik})^2]^{1/2}} \quad (13)$$

Then, these amplitudes A_{ik} are normalized in the form

$$\sum_i l_{ik}^2 = 1 \quad (14)$$

To fulfil a physical requirement, it is necessary to insert in Eq (13),

$$A_{ik} = L_k l_{ik} \quad (15)$$

where the L_k are constants determined by the initial values of the coordinates q_i and its time derivatives \dot{q}_i . This relations are substituted into the equation (9), the result being

$$q_i = \sum_{k=1}^N l_{ik} L_k \cos (\lambda_k^{1/2} t + \alpha_k). \quad (16)$$

In order to carry out the dynamical treatment of vibration, it is necessary to introduce a new set of coordinates Q_j , $j=1,2,\dots, N$, called normal coordinates. The normal coordinates are defined in terms of the displacement coordinates q_i by the linear equations

$$Q_j = \sum_{i=1}^N l_{ji}' q_i \quad j = 1, 2, \dots, N \quad (17)$$

in which the coefficients l_{ji}' must be chosen so as to be the kinetic and potential energies as follows,

$$2T = \sum_{j=1}^N \dot{Q}_j^2, \quad 2V = \sum_{j=1}^N \lambda_j' Q_j^2 \quad j = 1, 2, \dots, N. \quad (18)$$

Therefore, the potential energy in terms of the normal coordinates express no cross products but only squares of Q 's, and the kinetic energy maintains its original form. When normal coordinates are used, the equations of motion are given by

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{Q}_j} + \frac{\partial V}{\partial Q_j} = \ddot{Q}_j + \lambda_j' Q_j = 0 \quad j = 1, 2, \dots, N \quad (19)$$

The solutions of which are

$$Q_j = B_j \cos (\lambda_j^{1/2} t + \alpha_j) \quad j = 1, 2, \dots, N \quad (20)$$

where B_j and α_j are arbitrary constants. The relations between the q 's and Q 's must be satisfied the algebraic requirements of linear transformation. These circumstances should be verified by an accumulation of the knowleges of the mechanical vibration.

An Example

An artificial example is introduced to be helpful in illustrating the idea of normal vibration and normal coordinates. A linear system of two masses and three weightless springs provides an example as shown in Fig. 1. Masses m_1 and m_2 are connected to w_1 and w_2 through springs having spring constants f_1 and f_2 , and are connected together by a spring having spring constant f_{12} . The system is fixed at the points w_1 and w_2 . Therefore, no question of rotation and translation enters into the vibration under consideration, only linear

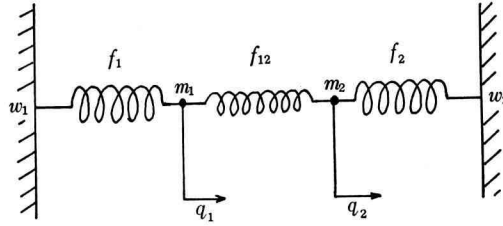


Fig. 1 Schematic diagram of the linear two-body vibrator.

motions of the two point masses. Only two coordinates are to be considered. They are called q_1 and q_2 , of which the displacements of mass points m_1 and m_2 are the quantities q_1 and q_2 from their equilibrium positions, respectively. To get the potential energy, the relations between f 's and q 's are used as follows

$$2V = f_1 q_1^2 + f_{12} q_{12}^2 + f_2 q_2^2 \quad (21)$$

in which q_{12} is the extension of the spring from the equilibrium length.

The kinetic energy of the whole system is simply

$$2T = m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2 \quad (22)$$

Then, the equations of motion of the two masses are given by

$$m_1 \ddot{q}_1 + (f_1 + f_{12}) q_1 - f_{12} q_2 = 0 \quad (23)$$

$$m_2 \ddot{q}_2 - f_{12} q_1 + (f_2 + f_{12}) q_2 = 0$$

The general solution can be written in the forms

$$q_1 = A_1 \cos (\lambda^{1/2} t + \alpha) \quad (24)$$

$$q_2 = A_2 \cos (\lambda^{1/2} t + \alpha)$$

By substitution, the Eq. (23) leads to the following two expressions

$$\begin{aligned} (f_1 + f_{12} - m_1 \lambda) A_1 - f_{12} A_2 &= 0 \\ -f_{12} A_1 + (f_2 + f_{12} - m_2 \lambda) A_2 &= 0 \end{aligned} \quad (25)$$

where the amplitudes A_1 and A_2 are the arbitrary constants. Above algebraic equations lead the determinant from their coefficients as

$$\begin{vmatrix} \frac{f_1 + f_{12}}{m_1} - \lambda & -\frac{f_{12}}{m_1} \\ -\frac{f_{12}}{m_2} & \frac{f_2 + f_{12}}{m_2} - \lambda \end{vmatrix} = 0 \quad (26)$$

This determinant is a quadratic equation in λ . From Eq (10), it means the frequency of vibration of the system. If $m_1 = m_2 = m$, and $f_1 = f_2 = f$, we get

$$\begin{aligned} \lambda_1 &= \frac{1}{m} (f + 2f_{12}) \\ \lambda_2 &= \frac{f}{m} \end{aligned} \quad (27)$$

Then, Eq. (25) yields the values of A_i

$$\begin{aligned} A_1 &= -A_2 \text{ for } \lambda = \frac{1}{m} (f + 2f_{12}) \\ A_1 &= A_2 \text{ for } \lambda = \frac{f}{m} \end{aligned} \quad (28)$$

The modes of vibration give the fundamental ones. For the first, the two mass points oscillate with the same frequency and equal amplitude. While the other oscillations have a opposite phases. The central spring does not enter during the motion because of lacking f_{12} in λ 's.

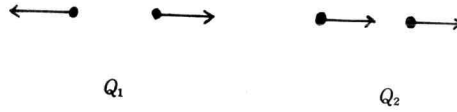


Fig. 2 Normal modes of vibration of the linear two-body vibrator for the special case, $m_1=m_2$ and $f_1=f_2$.

If we use the mass-weighted coordinates, $q_1' = m^{1/2}q_1$ and $q_2' = m^{1/2}q_2$. The kinetic and potential energies are given by

$$\begin{aligned} 2T &= \dot{q}_1'^2 + \dot{q}_2'^2 \\ 2V &= \frac{f_1 + f_{12}}{m_1} q_1'^2 - \frac{2f_{12}}{(m_1 m_2)^{1/2}} q_1' q_2' - \frac{f_2 + f_{12}}{m_2} q_2'^2 \end{aligned} \quad (29)$$

so that the determinant corresponding to Eq. (26) is written in the form

$$\begin{vmatrix} \frac{f_1 + f_{12}}{m_1} - \lambda & -\frac{f_{12}}{(m_1 m_2)^{1/2}} \\ -\frac{f_{12}}{(m_1 m_2)^{1/2}} & \frac{f_2 + f_{12}}{m_2} - \lambda \end{vmatrix} = 0 \quad (30)$$

The roots λ do not change, but the form of the determinant is symmetric. From the relations of Eqs. (11) and (13), the coefficient l_{ik} are determined by

$$\begin{aligned} \left[\frac{f_1 + f_{12}}{m_1} - \lambda_i \right] l_{1i} - \frac{f_{12}}{(m_1 m_2)^{1/2}} l_{2i} &= 0 \\ -\frac{f_{12}}{(m_1 m_2)^{1/2}} l_{1i} + \left[\frac{f_2 + f_{12}}{m_2} - \lambda_i \right] l_{2i} &= 0 \end{aligned} \quad (31)$$

The above equations determine the ratios of the l 's. Therefore, it can be given their absolute value by the normalization as follows

$$l_{11}^2 + l_{21}^2 = 1, \quad l_{12}^2 + l_{22}^2 = 1 \quad (32)$$

If $f_1=f_2$ and $m_1=m_2$, we get the relations

$$l_{11} = l_{12} = l_{22} = -l_{21} = 2^{-1/2} \quad (33)$$

so that the normal coordinates are

$$Q_1 = \frac{1}{2^{1/2}} (q_1 + q_2)$$

$$Q_2 = \frac{1}{2^{1/2}} (q_1 - q_2)$$
(34)

The kinetic and potential energies are given in the forms of normal coordinates

$$2T = \dot{Q}_1^2 + \dot{Q}_2^2$$

$$2V = \lambda_1 Q_1^2 + \lambda_2 Q_2^2$$
(35)

in which λ_1 and λ_2 are the two roots of the determinant transformed by the relations of Eq. (31). It is noticed that the values of two λ 's identified by Eq. (27) are related to the energy of the oscillator of the system which $\lambda = 4\pi^2\nu^2$. Further, the modes of vibration of the system are composed of several types of the infinitesimal displacement of the solid bodies.

Discussions

When a system undergone vibration is recorded its characteristic features by using a transducers such as accelerometers or strain gages, the record is characterized as being almost all sinusoidal one, which is called a simple harmonic motion. However, sometimes no obvious pattern in a vibration record is found. It is called a random vibration. Although the identical experiments are performed under the same conditions, the records continually differ from each other. These characteristic features are indicated in Fig. 3. The valuable informations of vibrations by Prof. Crandall⁸⁾, et. al., are related to the mechanical vibrations. The analyses on the wide and narrow band processes are physically considered to be the transformation between the time and frequency of vibrations by Fourier transform. The curves of spectral densities show the nature of vibrations. A simple and sharp peak of the density curve may indicate a neary harmonic vibration, while the wide

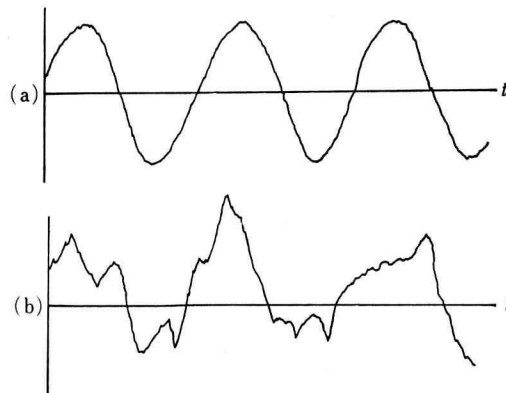


Fig. 3 Schematic diagram of the vibration records.
(a) sinusoidal; (b) random

band has to indicate a random vibration. The shapes of the density curve resemble to the infrared energy absorption bands of simple molecules. The narrow bands may correspond to the absorption bands of the symmetric and anti-symmetric bond stretching vibrations, on the other hand the wide bands may presumably show that of the bending mode of valence angle in molecules.

Prof. Timoshenko discusses the same example in his book. It is of much interest to analyze the vibrations in straightforward method using a matrix algebra. Much of valuable discussions are given in this book.

It is somewhat roundabout way to consider the theory of dynamics. The kinetic energy is given by in terms of the momenta

$$2T = \mathbf{p}^\dagger \mathbf{p} \quad (36)$$

where \mathbf{p}^\dagger is complex conjugate of transpose of the momentum matrix \mathbf{p} . If p_i is the momentum conjugate to q_i .

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = \dot{q}_i \quad (27)$$

so that the kinetic energy is expressed in terms of mass-weighted cartesian coordinates as

$$2T = \dot{\mathbf{q}}^\dagger \dot{\mathbf{q}} \quad (38)$$

where $\dot{\mathbf{q}}$ is the cartesian coordinate matrices corresponding to the displacements due to the vibrations of system. The cartesian coordinates of this displacement matrices are transformed into the normal coordinate matrices using a linear transformation. Especially, the application of matrix method is useful for containing a lot of mass points in a system. In these cases, it is extremely significant to solve the problems with a help of group theory.

Nomenclatures

- A : The amplitude of vibration.
- λ : The inverse of the frequency of vibration.
- ν : The arbitrary phase factor.
- p : The displacement coordinate of mass point from equilibrium position.
- Q : The normal coordinate.
- T : The kinetic energy of the system.
- V : The potential energy of the system.
- m : the mass of a particular point considered in this work.
- f : the spring constant connecting the mass point.

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