Stability Criteria for the System Represented by Half-Order Calculus

Makoto IKEDA

Department of Electrical Engineering

Abstract

This paper discusses stability criteria for the RC network containing distributed RC lines. This network is closely related to the non-integer integral and can be represented by $\sqrt{S}$ (S=differential operator). Though this network is practically important, this stability problem has not been investigated in the past owing to the presence of the non-integer operator $\sqrt{S}$ as the basic passive element. For this reason, in this paper, to begin with we will define the polynominal called the characteristic polynomial of $/S$ and we will show that the stability of this system is determined solely by the location of the roots of this polynominal in the 90° sector of $\sqrt{S}$-plane.

Thus, we wil find the algebraic condition for stability.

As the result, we will set up two criteria, Routh and Hurwitz types.

These criteria will be applied not only to this system but also to any other non-integer

1. Introduction

The non-integer integral operator is represented by $1/S^\nu$ (ν; integral order including non-integer, S=differential operator) from an operator theoretical point of view, and the systems expressed in terms of this operator, $1/S^\nu$, are usually called non-integer integral system or S^ν-parameter systems.

As Fig. 1 shows, we consider a single input-output system with the construction

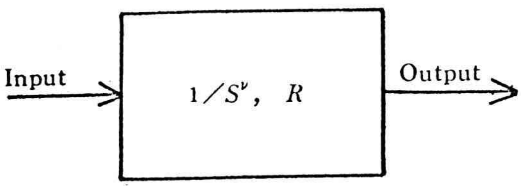

Fig. 1 $S\nu$I-System.

elements $1/S$I, R as constants.

The input-output relation of this system can be formulated by the following rational function with the variable S^ν.

$$F(S^\nu)=\frac{N(S^\nu)}{D(S^\nu)}=\frac{b_0(S^\nu)^{n'}+b_1(S^\nu)^{n'-1}+\cdots+bb_{n'}}{a_0(S^\nu)^{n}+a_1(S^\nu)^{n-1}+\cdots+a_n\cdot} \tag{1}$$

$$n'\leqq n$$

Received September 20 1978

In many problems of physical science that we encounter, this equation plays a paramount role, i.e.: heat conduction[1] (Carslaw and Jaeger, 1947), diffusion[2] (Babbitt, 1950, Crank, 1956), viscous flow[3] (Moore, 1964), neutron migration[4] (Davison, 1957), flow through porous media[5] (Muskat, 1937), the simplification of the high order differential equations, the complex control systems[6] (Manabe, 1960) and the realization ofnon-integer integral computing clements[7] (Hashimoto, 1969), to mention only a few.

In the field of network theory, the RC network comprising the uniform RC distributed lines can be thought of as this system, provided that $\nu=0.5$.
If the basis passive elements in this network are represented from the impedance point of view, these consist of three kinds; R, $1/SC$, $r\sqrt{S}$, and the impedance function or the transfer function of the transmission network constructed by these can be represented as the rational function $F(\sqrt{S})$ of one variale $\sqrt{S}$. Though this network is practically important in the construction of the thin film integrated circuit and so on, the stability problem of this network has not been investigated in the past owing to the presence of the non-integer operator $\sqrt{S}$ as the basis passive element.

The algebraic criteria in the conventional integer system, i.e. the lumped parameter system can not be adapted to this problem.

For this reason, we begin in this paper to define the polynominal called the characteristic polynominal of $\sqrt{S}$ and we will show that the stability of this system is determined solely by the loaction of the roots of this characteristic polynominal in the shadowed portion of $\sqrt{S}$ plane as shown in Fig. 2.

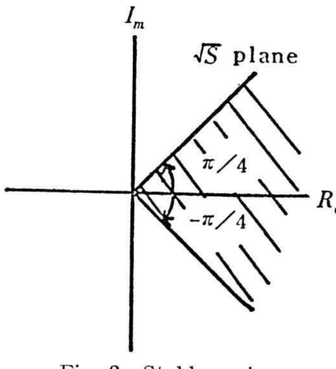

Fig. 2 Stable region.

Thus we will find the algebraic conditions for stability. As the result, we thus set up two criteria; the Routh and Hurwitz types. These criteria can be applied not only to this network which motivates us to think of the stability problem but also to the general non-integer integral systems[8] that arise in the many other diverse areas.

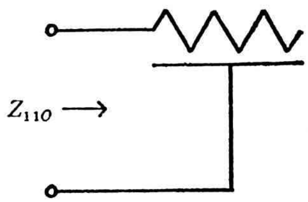

Fig. 3 (a) Open Termination.

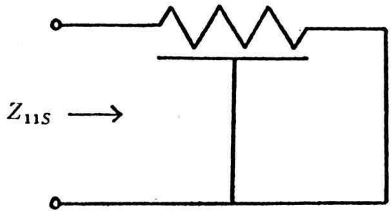

Fig. 3 (b) Close Termination.

2. $\sqrt{S}$ -parameter system

The driving point impedance of uniformly distributed RC line of length is represented in accorance with the following conditions.
In the case of open termination as shown in Fig. 3(a), we have

$$Z_{11O}=\frac{\gamma_0 \coth \gamma_0 L}{SC_0} \tag{2}$$

In the case of short termination as shown in ig. 3(b), we have

$$Z_{11S}=\frac{R_0}{\gamma_0 \coth \gamma_0 L} \tag{3}$$

where R_0 and C_0 are respectively resistance and capacitance of lines per unit length and $r_0 = \sqrt{SC_0 R_0}$ Now in (2) and (3), when $L \to \infty$, we obtain

$$Z_{11O},\ Z_{11S} \simeq \sqrt{\frac{R_0}{C_0}}\frac{1}{\sqrt{S}} \equiv \frac{r}{\sqrt{S}} \quad (S=jw) \tag{4}$$

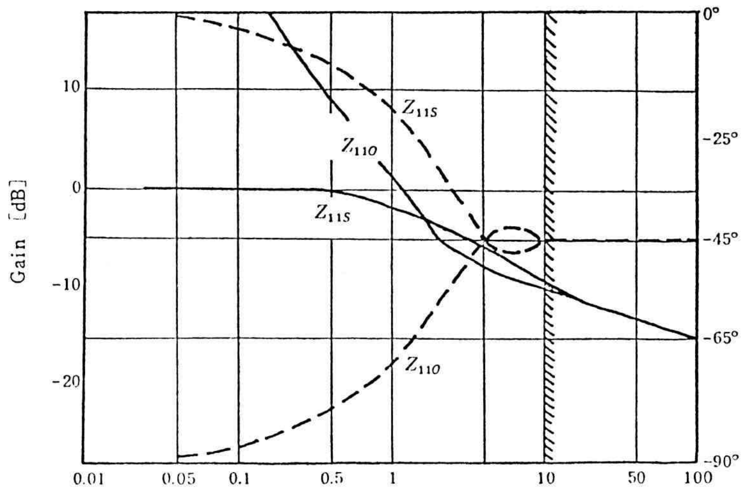

Fig. 3 (c) Phase characteristic, Amplitude characceristic.
(- - -) (——)

This indicates that the dryiving point impedance of the semi-infinite distributed RC line can be represented as $r/\sqrt{S}$.

The frequency characteristics of the amplitude and the phase with regard to Z_{110}, Z_{11s} in (2) and (3) are shown as in Fig. 3(c).
As apparant from this figure, even in the case of a finite length distributed line, deviation from the true value of Z_{110}, Z_{11s} is within one percent both in amplitude and phase characteristics over the angle frequency $w \geqq 10w_0$ provided that $w_0=1/C_tR_{t\,0}(C_t$, R_t represent the total capacitance and total resistance of the lines.)

Hence, the immittance functions or transfer functions of the system composed of lumped resistors, lumped capacitors and uniformly distributed RC lines can be approximately expressed as the real rational functions of the variable $\sqrt{S}$. We shall call these networks the $\sqrt{S}$-parameter systems and discuss its stability problem.

3. Pole location of $\sqrt{S}$-paramter system

As Fig. 4 shows, the voltage transmission function of the network composed of resisters, capacitors and infinite RC lines are described by the following expression:

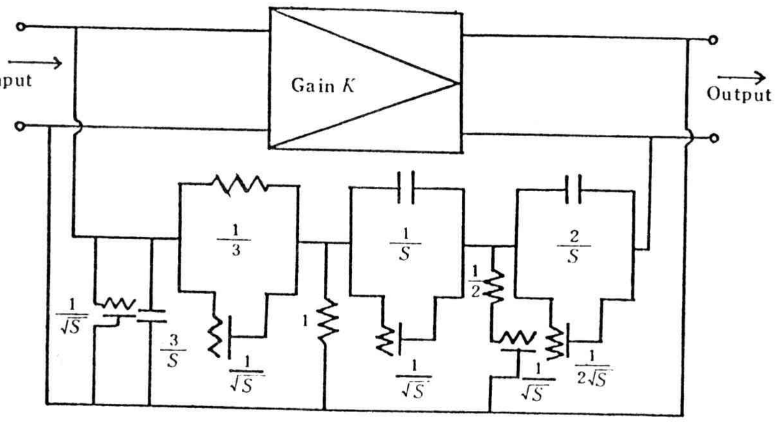

Fig. 4 The network composed of registers, capacitors and infinite RC lines.

$$F(\sqrt{S}) = K\frac{(\sqrt{S})^3+8(\sqrt{S})^2+19(\sqrt{S})+12}{(\sqrt{S})^3+(8-3K)(\sqrt{S})^2+(19-6K)\sqrt{S}+12} \tag{5}$$

Generally, RC networks including infinite RC distributed lines can be expressed as the real rational functions; $F(\sqrt{S})$ of the variable $\sqrt{S}$.
If we apply to $F(\sqrt{S})$ the method of partial fraction, we can express it in the following form.

$$F(\sqrt{S}) = \sum_{i=1}^{\zeta}\frac{\beta_i}{\sqrt{S}-\alpha_i} + \sum_{i=\zeta+1}^{\xi}\sum_{r=1}^{m_j}\frac{\beta_{jr}}{(\sqrt{S}-\alpha_j)^r} \tag{6}$$

Now we shall investigate the $\dfrac{1}{(\sqrt{S}-\lambda)^r}$ portion of Eq. (6).

Taking the inverse Laplace transform of $\dfrac{1}{(\sqrt{S}-\lambda)^r}$ we obtain

$$\mathscr{L}^{-1}\left[\frac{1}{(\sqrt{S}-\lambda)^r}\right]=\sum_m\sum_{i=1}^{r}A_{r-1}t^{r-1}e^{t|\lambda|^2\cos2(\theta_1+2m\pi)}e^{jt|\lambda|^2\sin2(\theta_1+2m\pi)}$$

$$+\frac{1}{2\pi_j}\int_0^\infty e^{-\cdot x}\frac{(x^{0.5}e^{j\frac{\pi}{2}}-\lambda)^r-(x^{0.5}e^{-j\frac{\pi}{2}}-\lambda)^r}{(x-2\lambda\cos\frac{\pi}{2}x^2+\lambda^2)^r}dx \tag{7}$$

where $\lambda=|\lambda|e^{j\theta_1}\neq 0,\ \theta_1=\arg\lambda\,(-\pi\leq\theta_1\leq\pi)$

$\sum\limits_m$ denotes the sum as long as integer m satisfying $-\dfrac{\pi}{2}<\theta_1+2m\pi<\dfrac{\pi}{2}$.

The second integral term of Eq. (7) can be easily thought of stable, since it becones zero when $t\to+\infty$. Next, we will consider the first term with regards to stability.

The first term gradually becomes smaller and smaller until it reaches zero when COS $2\ (\theta+2m\pi)<0$. Conversely when COS $2(\theta_1+2m\pi)\geq 0$, the first term increases monotonously to an unlimited extent or oscillates with increasing wave amplitude.

From the preceding discussion, we reach the following conclusion:

In order for the system function $F(\sqrt{S})$ in $\sqrt{S}$-parameter system to be stable it is necessary and sufficient that the denominator polynominal $D(\sqrt{S}($ has no roots in the shadowed region including the boundary of the $\sqrt{S}$ plane as shown in Fig. 2. Hence we call $D(\sqrt{S})$ the character- istic polynominal of $\sqrt{S}$.

Upon this conclusion, in the next section we will discuss the algebraic conditions needed in order for $D(\sqrt{S})$ to have no roots in the shadowed region including the boundary.

4. Stability criteria

4–1 The relation between the principle of the agrument and the Cauchy index

As Fig. 2 shows, since θ is not $\frac{1}{2}\pi$, we can not use the conventional relation between the Cauchy index and principle of the argument. Therefore, we need to reconsider its fundamentals.

When, for convenience, we make the substitution $\sqrt{S}=z$, the denominator polynominal of $F(\sqrt{S})$: $D(\sqrt{S})$ becomes

$$D(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$$

$$a_0>0,\ a_n\neq 0,\ a_i;\ \text{the real number}$$

From the conclusion of the third section, in order for $F(\sqrt{S})$ to be stable, it is necessary and sufficient that the polynominal $D(z)$ has no roots within the shadowed region Γ_3 Γ_4 Γ_5 including the boundary of the z-plane as shown in Fig. 5. We assume that $D(z)$ has

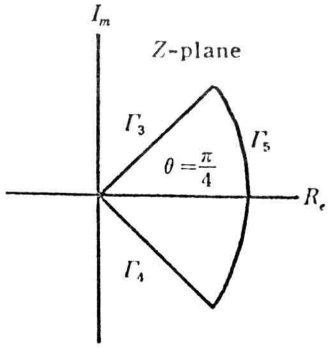

Fig. 5

k roots within the region $\Gamma_3\,\Gamma_4\,\Gamma_5$ and $2h$ roots on the boundaries Γ_3 and Γ_4.
From this assumption.

$$\begin{aligned} D(z) &= H(z)\tilde{H}(z)D^{**}(z) = H(z)D'(z) \\ D'(z) &= \tilde{H}(z)D^{**}(z) \end{aligned} \tag{8}$$

$\sim$ means complex conjugate,
where the polynominal of the roots on Γ_3 is

$$H(z) = (z-s_1)(z-s_2)\cdots(z-s_h)$$

and the polynominal of the roots on Γ_4 is

$$\tilde{H}(z) = (z-\tilde{s}_1)(z-\tilde{s}_2)\cdots(z-\tilde{s}_h)$$

When we express $D(z)$ in polar form $z=r\cdot e^{j\theta}\left(\theta=\frac{\pi}{4}\right)$, $D(z)$ is denoted by $D^{\theta}(r)$ and Eq. (8) can thus be transformed into:

$$\begin{aligned} D(r\cdot e^{j\theta}) &= D_\theta(r) = H_\theta(r)\tilde{H}_\theta(r)D_\theta^{**}(r) = H_\theta(r)D'_\theta(r) \\ &= D_{\theta R}(r) + jD_{\theta I}(r) \end{aligned} \tag{9}$$

$$D'(r\cdot e^{j\theta}) = D'_\theta(r'_{\theta R}) = D'_{\theta I}(r) \tag{10}$$

$$H_\theta(r) = e^{jh\theta}(r-r_1)(r-r_2)\cdots(r-r_h) = e^{jh\theta}f(r)$$

$$\text{where } s_1=r_1e^{j\theta},\ s_2=r_2e^{j\theta},\ \cdots,\ s_h=r_he^{j\theta}$$

$$D_\theta(r) = H_\theta(r)D'_\theta(r) = f(r)e^{jh\theta}D'_\theta(r) = f(r)D_\theta^*(r) \tag{11}$$

$$D_\theta^*(r) = D_{\theta r}^*(r) + jD_{\theta I}^*(r) \tag{12}$$

$$\begin{aligned} D_{\theta r}^*(r) &= \cosh\theta\cdot D'_{\theta r}(r) - \sinh\theta\cdot D'_{\theta I}(r) \\ &= c_0^*r^{n-h} + c_1^*r^{n-h-1} + \cdots + c_{n-h}^* \\ D_{\theta I}^*(r) &= \sinh\theta\cdot D'_{\theta r}(r) + \cosh\theta\cdot D'_{\theta I}(r) \\ &= d_0^*r^{n-h} + d_1^*r^{n-h-1} + \cdots + d_{n-h-2}^*r \end{aligned}$$

where c_0^*, c_0^*, $\cdots$, c_{n-h}^*, d_0^*, d_1^*, $\cdots$, d_{n-h-1}^* ; real constant.

$$\begin{aligned} D_{\theta r}(r) &= a_0\cos n\theta\cdot r^n + a_1\cos(n-1)\theta\cdot r^{n-1} + \cdots + a_n \\ &= c_0r^n + c_1r^{n-1} + \cdots + c^1_{-1}r + c_n \\ D_{\theta I}(r) &= a_0\sin n\theta\cdot r^n + a_1\sin(n-1)\theta\cdot r^{n-}{}_n + \cdots + a_{n-1}\sin\theta\cdot r \\ &= d_0r^n + d_1r^{n-1} + \cdots + d_{n-1}r \end{aligned} \tag{13}$$

where $c_i=a_i\cos(n\text{-}i)\theta$, $d_i=a_i\sin(n\text{-}i)\theta$.

In Eqs. (9) and (10), with r varying from $+\infty$ to 0 along Γ_3, the increase of $\arg D_0(r)$, $\underset{r}{\Delta^0_\infty} \arg D_\theta(r)$ is equal to the increase of $\arg D_\theta{}^*(r)$ as long as we ignore h roots on Γ_3. With r varying from $+\infty$ to 0, $\arg D_\theta{}^*(r)$ increases by $k(\pi\text{-}2\theta)$ owing to k roots within $\Gamma_3\Gamma_4\Gamma_5$ and by $h(\pi\text{-}2\theta)$ owing to h roots on Γ_4 on the other hand decreases by $(n\text{-}k\text{-}2h)\theta$ owing to $(n\text{-}k\text{-}2h)$ roots outside $\Gamma_3\Gamma_4\Gamma_5$. Hence, the result is

$$\underset{r}{\Delta^0_\infty} \arg D^*_\theta(r) = k\pi + h\pi - n\theta \tag{14}$$

Next, we consider the relation between $\underset{r}{\Delta^0_\infty} \arg D^*_\theta(r)$ and the Cauchy index $I^0_\infty \dfrac{D^*_{\theta I}(r)}{D^*_{\theta r}(r)}$ in accordance with the conditions of $\cos_n 0$.

(i) $\cos n\theta \neq 0,\ c_n \neq 0$

In this case, the following relation is established: (Proof is shown in the Appendix.)

$$\frac{1}{\pi}\{\underset{r}{\Delta^0_\infty} \arg D^*_\theta(r) + n\theta\} = -I^0_\infty \frac{D^*_{\theta I}(r)}{D^*_{\theta r}(r)} + \eta$$

$$\eta = \left[\frac{1}{2}\left(\frac{2n\theta}{\pi} + 1\right)\right],\ \theta = \frac{\pi}{4} \tag{15}$$

where, $D(r)$, $D^*(r)$ are assumed to hae no common factor, and ( ) is Gauss notation. From Eqs. (14) and (15), we obtain the following formula.

$$k + h = -I^0_\infty \frac{D^K_{\theta I}(r)}{D^*_{\theta r}(r)} + \eta \tag{16}$$

(ii) $\cos n\theta = 0,\ c_1 = \cdots = c_{m-1} = 0$, and $c_m \neq 0\ (m \geqq 1)$

In this case, the following relation is established: (Proof is shown in the Appendix.)

$$\frac{1}{\pi}\underset{r}{\Delta^0_\infty} \arg D^*_\theta(r) = -I^0_\infty \frac{D^*_{\theta I}(r)}{D^*_{\theta r}(r)} \mp \frac{1}{2} \quad (d_0 c_m \gtrless 0) \tag{17}$$

where $D_{\theta I}{}^*(r)$ and $D_{\theta r}{}^*(r)$ are assumed to have no common factor.
From Eqs. (14) and (17), we obtain the following formula.

$$k + h = I^\infty_0 \frac{D^*_{\theta I}(r)}{D^*_{\theta r}(r)} + \frac{n\theta}{\pi} \pm \frac{1}{2} \quad (d_0 c_m \gtrless 0) \tag{18}$$

4–2 The Cauchy index and Extended Routh's algorithm

In this section we find the algebraic algorithm for stability criteria, using Sturn's theorem and Cauchy indices using the classical theorem of Sturm.

(i) $\mathrm{con}\, n\theta \neq 0,\ c_n \neq 0$

From Eq. (13),

$$I^\infty_0 \frac{D_{\theta I}(r)}{D_{\theta r}(r)} = I^\infty_0 \frac{d_0 r^n + d_1 r^{n-1} + \cdots + d_{n-1} r}{c_0 r^n + c_1 r^{n-1} + \cdots + c_m} \tag{19}$$

Now we make the following substitution.

$$I^\infty_{-\infty} \frac{D_{\theta I}(r^2)}{rD_{\theta r}(r^2)} = I^\infty_{-\infty} \frac{d_0 r^{2n} + d_1 r^{2n-2} + \cdots + d_{n-1} r^2}{c_0 r^{2n+1} + c_1 r^{2-1} + \cdots + c_n r} \tag{20}$$

Thus, we obtain

$$I_0^{\infty}\frac{D_{\theta I}(r)}{D_{\theta r}(r)}=\frac{1}{2}I_{-\infty}^{\infty}\frac{D_{\theta I}(r^2)}{rD_{\theta r}(r^2)} \tag{21}$$

Therefore, the Cauchy index of Eq. (19) is equal to half that of Eq. (2), where from Eq. (8),
$D_{\theta I}(r)$ and $D_{\theta r}(r)$ have h common factors and
$D_{\theta I}(r^2)$ and $D_{\theta r}(r^2)$ have $2h$ common factors.

We apply Sturm's theorem in the interval $(-\infty, +\infty)$ to the right side of Eq. (20). We set

$$\begin{aligned} f_0(r) &= c_0 r^{2n}-(-c_1)r^{2n-2}+c_2 r^{2n-4}\cdots \\ f_1(r) &= d_0 r^{2n-1}-(-d_1)r^{2n-3}+d_2 r^{2n-5}-(-d_3)r^{2n-7}\cdots \end{aligned} \tag{22}$$

From these two polynominals, we can construct a generalized Sturm chain $f_0, f_1 \ldots.$ by the Euclidean algorithm.

Now, we denote by $V(x)$ the number variations of sign in the chain with a fixed value of x.
In Sturm's theorem,

$$I_{-\infty}^{\infty}\frac{D_{\theta I}(r^2)}{rD_{\theta r}(r^2)}=V_{2n}(-\infty)-V_{2n}(+\infty) \tag{23}$$

and since $D_{\theta I}(r^2)$ and $D_{\theta r}(r^2)$ have $2h$ common factors

$$V_{2n}(-\infty)+V_{2n}(+\infty)=2n-2h \tag{24}$$

Since $VV_{2n-2h+1}=\cdots=V_{2n}=0$, from Eqs. (23) and (24),

$$I_{-\mp}^{\infty}\frac{D_{\theta I}(r^2)}{rD_{\theta r}(r^2)}=2\{n-h-V_{2n-2h}(+\infty)\} \tag{25}$$

From Eqs. (21) and (25), we obtain

$$I_0^{\pm}\frac{D_{\theta I}(r)}{rD_{\theta r}(r)}=n-h-V_{2}{}^{u}{}_{-2h}(+\infty) \tag{26}$$

Since $D_{\theta I}(r)$ and $D_{\theta r}(r)$ have h common factors, Eq. (26) becomes

$$I_0^{\pm}\frac{D_{\theta I}(r)}{D_{\theta r}(r)}=I_0^{\pm}\frac{D_{\theta I}^*(r)}{D_{\theta r}^*(r)}=n-h-V_{2n-2h}(+\infty) \tag{27}$$

(ii) $\cos n\theta=0,\ c_1=\cdots c_{m-1}=0,\ c_m\neq 0\,(m\geq 1)$

In this case,

$$\begin{aligned}\frac{D_{\theta I}(r)}{D_{\theta r}(r)} &= \frac{d_0 r^n+d_1 r^{n-1}+\cdots+d_{n-1}r}{c_m r^{n-m}+c_{m+1}r^{n-m-1}+\cdots+c_{n-1}r+c_n} \\ &= E_m(r)+\frac{d'_m r^{n-m}+d'_{m+1}r^{n-m-1}+\cdots+d'_{n-1}r}{c_m r^{n-m}+c_{m+1}r^{n-m-1}+\cdots+c_{n-1}r+c_n}\end{aligned}$$

where $E_m(r)$ is the quotient.

We make the following substitution.

$$I_{-\mp}^{\pm}\frac{D_{\theta I}(r^2)}{rD_{\theta r}(r^2)}=I_{-\mp}^{\pm}\frac{d'_m r^{2n-2m-1}+d'_{m+1}r^{2n-2m-3}+\cdots+d'_{n-1}r}{c_m r^{2n-2m}+c_{m+1}r^{2n-2m-2}+\cdots+c_{n-1}r^2+c_n} \tag{28}$$

Thus, we obtain,

$$I_0^{\pm}\frac{D_{\theta I}(r)}{D_{\theta r}(r)}=\frac{1}{2}I_{-\mp}^{\pm}\frac{D_{\theta I}(r^2)}{rD_{\theta r}(r^2)} \tag{29}$$

From Eq. (29), we obtain

$$I_0^{\pm}\frac{D_{\theta I}(r)}{D_{\theta r}(r)}=I_0^{\pm}\frac{D^*_{\theta I}(r)}{D^*_{\theta I}(r)}=n-m-h-V_L(+\infty) \tag{30}$$

$$L=2n-2m-2h$$

From the proceeding discussion, we obtain

(i) $\cos n\theta \neq 0,\ c_n \neq 0$

From Eqs. (16) and (27)

$$k=n-2h+\eta-V_{2n}(+\infty)$$
$$\eta=\left[\frac{1}{2}\cdot\left(\frac{2n\theta}{\pi}+1\right)\right] \tag{31}$$

We need the following extended Routh's table when using Eq. (31).

$$\begin{array}{l} r_{00}=a_0\cos n\theta,\quad r_{01}=-a_1\cos(n-1)\theta,\ \cdots,\ r_{0n} \\ r_{11}=a_0\sin n\theta,\quad r_{12}=-a_1\sin(n-1)\theta\ \cdots,\ r_{1n} \\ r_{22}\qquad\qquad\quad , \\ \vdots \\ r_{LL},\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\ r_{Ln} \end{array} \tag{32}$$

where $L=2n\text{-}2h$, every row in this table is determined by the two preceding rows according to the following rule.

$$r_{ij}=\frac{r_{i-1,\,i-1}\cdot r_{i-2,\,j-1}-r_{i-2,\,j-2}\cdot r_{i-1,\,j}}{r_{i-1,\,i-1}} \tag{33}$$

and $V(\cdots\cdot)$ is the number of variations of sign in the first column of Routh's table.

(ii) $\cos n\theta=0,\ c_1=\cdots=c_{m-1}=0,\ c_m\neq 0,\ (m\geq 1)$

From Eqs. (18) and (30)

$$k=n-m-2h+\frac{n\theta}{\pi}\pm\frac{1}{2}-V_L(+\infty)\quad d_0c_m\gtreqless 0 \tag{34}$$

$$L=2n-2m-2h$$

We need the following extended Routh's table when using Eq. (34)

$$\begin{array}{l} \gamma_{00}=a_m\cos(n-m)\theta,\ \gamma_{01}=-a_{m+1}\cos(n-m-1)\theta,\ \cdots,\ \gamma_{0n} \\ \gamma_{11}=d'_m\qquad\qquad\ ,\ \gamma_{12}=-d'_{m+1},\ \cdots\cdots\cdots,\ \cdots,\ \gamma_{1n} \\ \vdots\qquad\qquad\qquad\quad \vdots \end{array}$$

where every row in this table is determined by the two proceeding rows according to the same rule of Eq. (33) and $d'_m,\ d_{m+1},\cdots$ corresponds to each coefficient of the remainder after dividing the imaginary part by the real one of $D_\theta(r)$ by way of the Euclidean algorithm. Thus, summarizing we have the following theorems.

[Theorem 1]

When $\cos n\theta\neq 0,\ a_n\neq 0$, the system (1) is stable if and only if the next Eq. (36) holds

$$n+\eta=V_{2n}(r_{00},\ r_{11},\ \cdots,\ r_{2n,\,2n})$$
$$\eta=\left[\frac{1}{2}\left(\frac{2n\theta}{\pi}+1\right)\right] \tag{36}$$

where $\theta = \frac{\pi}{4}$

[Theorem 2]

When $\cos n\theta = 0,\ a_n \neq 0,\ a_1 \operatorname{ccs}(n-1)\theta = \cdots = a_{m-1} \operatorname{oos}(n-m+1)\theta = 0,\ a_m \cos(n-m)\theta \neq 0 (m \geq 1)$, if and only if next Eq. (37) holds, the system (1) is stable.

$$n - m + \frac{n\theta}{\pi} \pm \frac{1}{2} = V_L(\gamma_{00}, \gamma_{11}, \cdots)\quad a_0 \sin n\theta \cdot a_m \cos(n-m)\theta \gtrless 0$$

$$L = 2n - 2m,\quad \theta = \frac{\pi}{4} \tag{37}$$

4–3 The Cauchy index and it's determinant form

In this section, we describe the Cauchy Index in (16) and (18) by the determinant form whose elements are composed of the coefficients $D(z)$.

Prior to doing so, we need first to state the next theorem[9].

[Theorem 3]

Ket

$$R(z) = \frac{h(z)}{g(z)} = \frac{\alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n}{\beta_0 z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta^n} \tag{38}$$

be a proper rational function. ($a_0 \neq 0,\ a_i,\ p_i$; real number) If $g(z)$ and $h(z)$ are relatively prime, the Cauchy index $I_{-\infty}^{\infty} \frac{g(z)}{h(z)}$ can be determined by the following formula.

where Δ_{2p} is the $2p$ order principal minor of the following determinant.

$$I_{-\infty}^{\infty} \frac{g(z)}{h(z)} = n - 2V(1, \Delta_2, \Delta_4, \cdots, \Delta_{2n}) \tag{39}$$

$$\Delta_{2n} = \begin{vmatrix} \alpha_0, & \alpha_1, & \alpha_2, & \cdots\cdots \\ \beta_0, & \beta_1, & \beta_2, & \cdots\cdots \\ 0\ , & \alpha_0, & \alpha_1, & \cdots\cdots \\ 0\ , & \beta_0, & \beta_1, & \cdots\cdots \\ \cdot\ , & \cdot\ , & \cdot\ , & \cdots\cdots \\ \cdot\ , & \cdot\ , & \cdot\ , & \cdots\cdots \end{vmatrix} \tag{40}$$

and $V(1, \Delta_2, \Delta_4, \cdots, \Delta_{2n})$ is the number of variations of sign in the sequence $1, \Delta_2, \Delta_4, \cdots, \Delta_{2n}$. If $g(z)$ and $h(z)$ are not relatively prime and have L common factors, then

$$I_{-\infty}^{\infty} \frac{g(z)}{h(z)} = n - L - 2V(1, \Delta_2, \Delta_4, \cdots, \Delta_{2(n-L)}) \tag{41}$$

Using this theorem, we describe the Cauchy index of (16) and (18) in the determinant form.

In the same manner as 4–2, we consider it in accordance with the conditions of $\cos {}_n\theta$.

(i) $\cos {}_n\theta \neq 0,\ c_n \neq 0$

In order to use theorem 3, we need the relation in (21). Eq. (21) is further transformed into the following:

$$\frac{1}{2} I_{-\infty}^{\infty} \frac{D_{\theta I}(r^2)}{r D_{\theta r}(r^2)} = - I_{-\infty}^{\infty} \frac{D_{\theta I}(r^2)}{D_{\theta r}(r^2)} = - I_{-\infty}^{\infty} \frac{f_{\theta I}(-r^2)}{f_{\theta I}(-r^2)} = - I_{\mp}^{\infty} \frac{f_{\theta I}(u)}{f_{\theta r}(u)}$$

$$= \frac{1}{2}\left(I_{\mp}^{\pm} \frac{f_{\theta I}(u)}{u f_{\theta r}(u)} - I_{\mp}^{\pm} \frac{f_{\theta I}(u)}{f_{\theta r}(u)} \right) \tag{42}$$

where $f_{\theta r}(u) = c_0 u^n - c_1 u^{n-1} + \cdots + (-1)^n c_n$

$f_{\theta I}(u) = d_0 u^n - d_3 u^{n-1} + \cdots + (-1)^n d_n$

$c_0, c_1, \cdots, c_n, d_0, d_1, \cdots, d_n$ are given by Eq. (13).

If we apply theorem 3 to the second term in (42), we obtain

$$I_{\mp}^{\pm} \frac{f_{\theta I}(u)}{f_{\theta r}(u)} = n - 2V(1, B_2, B_4, \cdots, B_{2n}) \tag{43}$$

B_{2P} is the $2p$-order principal minor of the following determinant.

$$B_{2n} = \begin{vmatrix} c_0, & -c_1, & c_2, & & ,0 \\ d_0, & -d_1, & d_2, & & \\ 0\,, & c_0, & -c_1, & & \\ 0\,, & d_0, & -d_1, & & \\ \vdots & & & & \\ 0\,, & & & & ,(-1)^n d_n \end{vmatrix} \tag{44}$$

On the other hand, since in the first term of (42), d_n vanishes, $f_{\theta I}(u)$ and $u f_{\theta r}(u)$ have a common factor u.

Therefore, using theorem 3, we obtain

$$I_{\mp}^{\pm} \frac{f_{\theta I}(u)}{u f_{\theta r}(u)} = n - 2V(1, C_2, C_4, \cdots, C_{2n}) \tag{45}$$

where C_{2P} is $2p$-order principal minor of the following determinant.

$$C_{2n} = \begin{vmatrix} c_0, & -c_1, & c_2, & -c_3, & & ,0 \\ 0\,, & d_0, & -d_1, & d_2, & & \\ 0\,, & c_0, & -c_1, & c_2, & & \\ 0\,, & 0\,, & d_0, & -d_1, & & \\ \vdots & & & & & \\ 0\,, & & & & & ,(-1)^n d_{n-1} \end{vmatrix} \tag{46}$$

Though determinants B and C are different, they can be respectively related to the following determinant E.

$$E_{2n} = \begin{vmatrix} d_0, & -d_1, & d_2, & & ,0 \\ c_0, & -c_1, & c_2, & & \\ 0\,, & d_0, & -d_1, & & \\ 0\,, & c_0, & -c_1, & & \\ \vdots & & & & \\ 0\,, & & & & ,(-1)^n c_n \end{vmatrix} \tag{47}$$

The relations between determinants B, C and E are

$$(-1)^p E_{2p} = B_{2p}, \quad C_0 E_{2p-1} = C_{2p} \tag{48}$$

Thus using this relation in Eqs. (43) and (45). Thus, we obtain

$$I^{\pm}_{-\mp} \frac{f_{\theta I}(u)}{f_{\theta r}(u)} = -n + h + 2V(1, E_2, E_4, \cdots, E_{2n}) \tag{49}$$

$$I^{\pm}_{-\mp} \frac{f_{\theta I}(u)}{uf_{\theta r}(u)} = n - h - 2V(C_0, E_1, E_3, \cdots, E_{2n-1}) \tag{50}$$

From (21), (42), (49) and (50)

$$I^{\pm}_{0} \frac{D_{\theta I}(r)}{D_{\theta r}(r)} = n - V(C_0, E_1, E_2/E_1, \cdots, E_{2n}/E_{3n-1}) \tag{51}$$

The number of roots of the polynominal $D(z)$ within $\Gamma_3\Gamma_4\Gamma_5$ including the boundary is determined by the following formula derived from (16) and (51).

$$k + h = n + \eta - V(C_0, E_1, E_2/E_1, \cdots, E_{2n}/E_{2n-1}) \tag{52}$$

Thus when $k=0$, $h=0$, the following theorem for stability criteria can be established.

[Theorem 4]

When $\cos n\theta \neq 0$, $a_n \neq 0$, if and only if the next Eq. (53) holds, the system (1) is stable.

$$n + \eta = V(C_0, E_1, E_2/E_1, \cdots, E_{2n}/E_{2n-1}) \tag{53}$$

where $\quad c_0 = a_0 \cos n\theta, \ \eta = \left[\frac{1}{2}\left(\frac{2n\theta}{\pi} + 1\right)\right], \ \theta = \frac{1}{4}\pi$

E_P is the $2p$-order principal minor of the determinant $E^2{}_n$ in Eq. (48).

(ii) $\cos n\theta = 0$, $c_1 = \cdots = c_{m-1} = 0$, $c_m \neq 0 \ (m \geq 1)$

The Cauchy index can be described by the following formula in much the same way as in (i)

$$I^{\pm}_{0} \frac{D_{\theta I}(r)}{D_{\theta r}(r)} = \frac{1}{2}\left(I^{\pm}_{-\mp} \frac{f_{\theta I}(u)}{uf_{\theta r}(u)} - I^{\pm}_{-\mp} \frac{f_{\theta I}(u)}{f_{\theta r}(u)}\right) \tag{54}$$

Now we denote by $R(u)$ the quotient and by $\tilde{f}_{\theta I}(u)$ the remainder after dividing $f_{\theta I}(u)u$ and $uf_{\theta r}(u)$, then $\tilde{f}_{\theta I}(u)$ is represented by the following

$$\tilde{f}_{\theta I}(u) = (-1)^m \tilde{d}_m u^{n-m} + (-1)^{m+1}\tilde{d}_{m+1} u^{n-(m+1)} + \cdots + (-1)^n \tilde{d}_n \tag{55}$$

and since $R(u)$ and $uR(u)$ are the polynominal of u not related to the Cauchy index Eq. (54) becomes

$$I^{\pm}_{0} \frac{D_{\theta I}(r)}{D_{\theta r}(r)} = \frac{1}{2}\left(I^{\pm}_{-\mp} \frac{\tilde{f}_{\theta I}(u)}{uf_{\theta r}(u)} - I^{\pm}_{-\mp} \frac{\tilde{f}_{\theta I}(u)}{f_{\theta r}(u)}\right. \tag{56}$$

If we apply theorem 3 to (55) in the same way as was done in (i), we obtain

$$I^{\pm}_{-\mp} \frac{\tilde{f}_{\theta I}(u)}{f_{\theta r}(u)} = -n - h - m) - 2V(1, B'_2, B'_4, \cdots, B'_{2(n-m)}) \tag{57}$$

$$I^{\pm}_{-\pm} \frac{\tilde{f}_{\theta I}(u)}{uf_{\theta r}(u)} = (n - h - m) - 2V(1, C'_2, C'_4, \cdots, C'_{2(n-m)}) \tag{58}$$

where the determinant B' is determined by the coeffecients of $f_{\theta r}(u)$ and $\tilde{f}_{\theta I}(u)$ and C' is also determined by that of $uf_{\theta r}(u)$ and $\tilde{f}_{\theta I}(u)$.

B' and C' are respectively related to the determinant E in (41), according to the following relations when

$c_0 = c_1 = \cdots = c_m = 0$

$$B'_{2p} = (-1)^{p} \frac{E_{2(p+m)}}{E_{2m}}, \quad C'_{2p} = \frac{E_{2(m+p)-1}}{E_{2m-1}} \tag{59}$$

Using these realtions, the equation (56) can be rewritten as follows.

$$I^{\pm}_{-\mp} \frac{D_{\theta I}(r)}{D_{\theta r}(r)} = (n - h - m) - V(E_{2m}/E_{2m-1}, E_{2m+1}/E_{2m}, \cdots, E_{2n}/E_{2n-1}) \quad d_0 c_m \gtrless 0) \tag{60}$$

By the following formula, derived from (18) and (60), the number of roots of the polynominal $D(z)$ within $\Gamma_3 \Gamma_4 \Gamma_5$ including the boundary is determined, thus:

$$k = (n - h - m) + \frac{n\theta}{\pi} \mp \frac{1}{2} - V(E_{2m}/E_{2m-1}, E_{2m+1}/E_{2m}, \cdots, E_{2n}/E_{2n-1}) \quad d_0 c_m \gtrless 0 \tag{61}$$

Thus when $k=0$, h=0, we obtain the following theorem for stability criteria from (61).

[Theorem 5]

In the case when $\theta \neq 0$, $a_n \neq 0$, $a_0 \cos n\theta = a_1 \cos(n-1)\theta, \cdots = a_{m-1} \cos(n-m+1)\theta = 0$, $a_m \cos(n-m)\theta \neq 0$, $(m \geq 1)$, if and only if next Eq. (62) holds, the system (1) is stable,

$$(n - m) + \frac{n\theta}{\pi} \mp \frac{1}{2} = V(E_{2m}/E_{2m-1}, E_{2m+1}/E_{2m}, \cdots, E_{2n}/E_{2n-1}) \quad (a_0 \sin n\theta \cdot a_m \cos(n-m)\theta \gtrless 0) \tag{62}$$

where $E^2{}_n = 0$ and $E^2{}_p$ is $2p$-order principal minor of the determinant $E^2{}_n$ in Eq. (48).

5. Example

In the network as shown in Fig. 4, we examine the stability in case the gain of the feedback amplifier K is 3 and 4.

When $K=3$, the denominator polynominal $D(\sqrt{S})$ os the voltage transmission function $F(\sqrt{S})$ is represented as follows.

$$D(\sqrt{S})|_{K=3} = (\sqrt{S})^3 - (\sqrt{S})^2 + \sqrt{S} + 12$$

Using theorem 1, we examine its stability. In this case,

$$n + \eta = 4, \quad V(\cdots) = 4$$

Hence when $K=3$, the system is stable.

Similarly, using theorem 3, we obtain the same result. When $K=4$, $D(\sqrt{S})$ is represented as follows.

$$D(\sqrt{S})|_{K=4} = (\sqrt{S})^3 - 4(\sqrt{S})^2 - 5(\sqrt{S}) + 12$$

Using theorem 1, we exmaine its stability. In this case,

$$n + \eta = 4, \quad V(\cdots) = 2$$

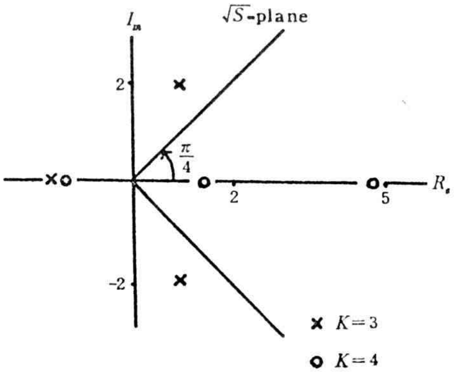

Fig. 6

Hence when $K=4$, the system is unstable.

Similarly, using theorem 3, we obtain the same result. We show the pole location in $\sqrt{S}$ plane in case $K=3$, and 4 in Fig. 6.

6. Conclusion

We have discussed stability critirea for the systemr epresented by $\sqrt{S}$. This result can be directly extended into the any other non-integer integral system represented by S^ν.

In this case, θ becomes $\frac{\pi\nu}{2}$ and we examined whether the denominator polynominal $D(S^\nu)$ has no roots in the shadowed region including the boundary of the S^ν-plane. The shadowed region differs according to the value of $\theta=\frac{\pi\nu}{2}$ and we can devide into it two cases as shown in Fig. 7.

Now we show some examples.

Example 1

The denominator polynominal $D(S^\nu)$ of the system function is given as follows.

$$D(S^{1/3})=(S^{1/3})^3-(S^{1/3})^2+2$$

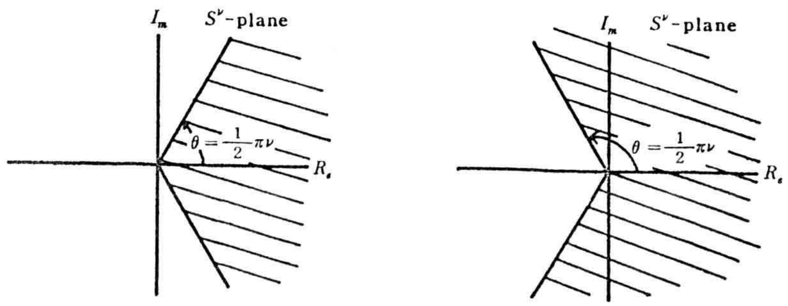

Fig. 7Stable region

Using theorem 2 or 5,

$$n - m + \frac{n\theta}{\pi} + \frac{1}{2} = 3 \qquad (d_0\, c_m < 0)$$

$$V\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{4}{3}\sqrt{3}, \frac{13}{4}, -2\right) = 3$$

Hence this system is stable.

Example 2

The denominator polynominal $D(S^{\nu})$ of the system function is given as follows.

$$D(S^{3/2}) = (S^{3/2})^3 - 3(S^{3/2})^2 + (S^{3/2}) + 5$$

Using Theorem 1 or 4,

$$n + \eta = 5$$

$$V\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 3, -\frac{8}{3}, \frac{\sqrt{2}}{2}, -12\sqrt{2}, -5\right) = 3$$

Hence this system is unstable and two roots lie inside the region between $\left[-\frac{3}{4}\pi, \frac{3}{4}\pi\right]$

We show the pole location of thoese examples in Fig. 8 and 9. If θ is above $\frac{\pi}{2}$ as shown

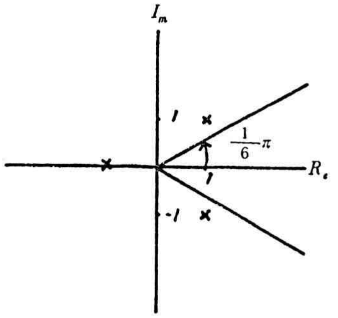

Fig. 8

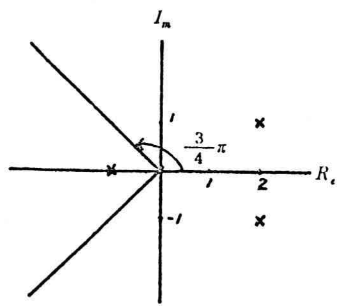

Fig. 9

in Fig. 9, without using these criteria the conventional criteria can be applied to the system.

The reason is if we made the substitution $z = r \cdot e^{j(\theta-\pi/2)}$ in the denominator polynominal $D(z)$ in the system (1), $D(re^{j(\theta-\pi/2)})$ bedomes the complex coefficient polynominal and we can examine whether it is the generalized Hurwitz one or not. This method is already suggested by E. Frank and D.D. Siljak. Nevertheless, these criteria set up by us here are the more valuable because it does not matter if θ is above, $\frac{\pi}{1}$, or not.

Appendix (Proof of (15) and (17))

Proof of (15)

Let us assume that $D^*_\theta(r)$ is plotted for r varying from $+\infty$ to 0 as shown in Fig. 10(a). From the standpoint of the increase of the Argument, the locus in Fig. 7(a) can be thought

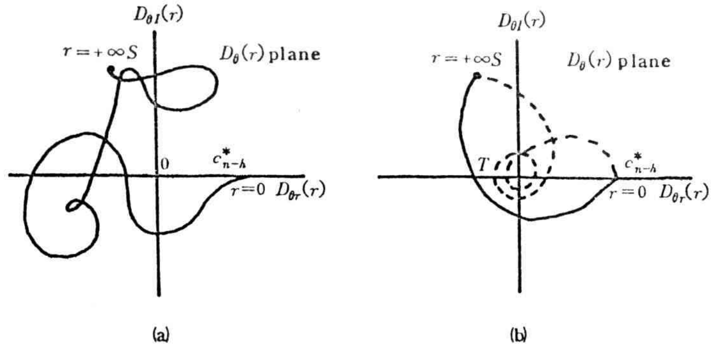

Fig. 10

to be equivalent to that in Fig. 10(b). Since the Cauchy index $I^0_{\mp}\dfrac{D^*_{\theta I}(r)}{D^*_{\theta r}(r)}$ is the difference between the numbers of jumps of $\dfrac{D^*_{\theta I}(r)}{D^*_{\theta r}(r)}$ from $+\infty$ to $+\infty$ and that of jumps from $+\infty$ to $+\infty$, it indicates also the difference between the number of crossings of the imaginary axis from the even number quadrant to the odd number one and vice versa in Fig. 10.

Therefore from the view point of the Cauchy index $I^0_{-\pm}\dfrac{D^*_{\theta I}(r)}{D^{*A}_{\theta}(r)}$, the locus in Fig. 10(a) is also equivalent to that in Fig. 10(b), so it doesn't matter if in Fig. 10(b) we can investigate the relations between the Cauchy index and the increase of the Argument.
If in Fig. 10(b), we denote the increase of the Argument between S and T by ξ.
Then we obtain

$$\frac{1}{\pi}\{\Delta^0_{r\pm}\arg D^*_\theta(r)-\xi\}=-I^0_-\frac{D^{*I}_\theta(r)}{D^*_{\theta r}(r)}$$

and when $r=+\infty$, $D^*_\theta(r)$ lies on the straight line of $\tan n\theta$, on the other hand, when lies at the point $C_{n-h}{}^*$ on the real axis $C^*{}_{n-h}>0$, in the cse of Fig. 10(b))

We consider the new locus expressed by the dotted line that rotates by $n\theta$ around the origin starting from the positive real axis and moving in the positive direction. Then we obtain,

$$\frac{1}{\pi}\{n\theta+\xi\}=\eta$$

where $$\eta=\left[\frac{1}{2}\left(\frac{2n\theta}{\pi}+1\right)\right]$$

From (55) and (56), Eq. (15) is derived.
Thus, we explain the case of Fig. 10(a).
This is established without losing generality except that $n\theta$ lies on the imaginary axis.
In the case of $n\theta$ lying on the imaginary axis, we use Eq. (17). Since this equation is easily proved as in Eq. (15), we shall omit its discussion here.

References

1) Carslaw, H.S., and Jaeger, J.C. "The Conduction of Heat in Solids," Oxford Univ. Press, London and New York; 1947.
2) Babbitt, J.D. "On the differential equations of diffusion", Canad. J. Res. Sect. A 28, 447; 1950.
3) Moore, F.K. "Theory of Laminar Flows," Princeton Univ. Press, Princeton, New Jersey; 1964.
4) Davison, B. "Neutron Transport Theory," Oxford Univ. Press, London and New York; 1957.
5) Muskat, M. "The Flow of Homogeneous Flnids tnrough Porous Media," McGraw-Hill, New York; 1937.
6) S. Manabe "The Non-inte erintegral and its Applicationto Control System" J.I.E.E.J·35–44; 1960.
7) S. Hasimoto, S. Takahashi and H. Amano "The Realization of Non-Integer Integral Computing Elements", J.I.E.E.J. 89–11;)969.
8) Oldham and Spanier "The Fractional Calculus," Academie press.
9) Gantmacher "Matrix Theory" Vol. II Chelsea.