Integral Representations of Scalar and Vector Potentials Characterizing the Electromagnetic Field due to Simple Harmonically Oscillating Sources Distributed in a Homogeneous, Isotropic and Conducting Medium

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Abstract

A theoretical analysis is made on the titled subject. First, using the Green's second thorem, the scalar potential and the magnetic scalar potential are derived as a solution of an inhomogeneous Helmholtz differential equation for the electric case and the magnetic case respectively. Secondly, using a result from the above analysis, the vector potential and the magnetic vector potential are derived as a solution of an inhomogeneous vector Helmholtz differential equation for the electric case and the magnetic case respectively. Thirdly, using the vector (analogue of) Green's second theorem, the same but in a different form of vector potential is derived as a solution of the same differential equation for either case electric or magnetic respectively. An identity of the two different forms of a vector potential which have been derived in different ways, is verified for the respective case. Finally, the radiation formulae and the Huygens principle in terms of the scalar potential and the vector potential are given for the electric case and the magnetic case. All solutions are given by an integral representation.

Introduction

First, the author derives from Maxwell equations the inhomogeneous Helmholtz and vector Helmholtz differential equations, which define the scalar and vector potentials in a simple harmonic state, respectively, for each type of electromagnetic field arising from electric or magnetic soruces.

Secondly, the author derives a solution in an integral form of the inhomogeneous Helmholtz differential equation, the integral itself being derived using Green's second theorem, for each case electric or magnetic. The result just mentioned is used for deriving a solution in an integral form of the inhomogeneous vector Helmholtz differential equation.

Thirdly, the author derives for each case electric or magnetic another solution in an integral form to the inhomogeneous vector Helmholtz differential equation using the vector (analogue of) Green's second theorem.

An identity of the above-mentioned two solutions which are derived in mutually different ways from the inhomogeneous vector Helmhotz differential equation is verified for each case.

Finally, a set of radiation formulae in terms of scalar and vector potentials are given by an integral representation, respectively, and the Huygens principle for each potential.

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In this paper, the SI unit system is used and a time factor $e^{j\omega t}$ is suppressed throughout. In addition, k[rad/m] and $\boldsymbol{a}$ are used as the propagation constant and the unit vector respectively. For the convenience of mathematical manipulation, formulae for vector analysis are given in Appendix.

1 Scalar potential and vector potential

1.1 Case of electric soruces

Consider the electromagnetic field due to the electric charge ρ_e [C/m³] and the electric current $\boldsymbol{J}$[A/m²] which are distributed in an infinite, homogeneous, isotropic and conducting medium. Maxwell's equations may then be written as

$$\nabla_Q\cdot\boldsymbol{E}=\frac{\rho_e}{\epsilon}, \tag{I}$$

$$\nabla_Q\cdot\boldsymbol{H}=0, \tag{II}$$

$$\nabla_Q\times\boldsymbol{H}=\boldsymbol{J}+(j\omega\epsilon+\sigma)\boldsymbol{E}, \tag{III}$$

$$\nabla_Q\times\boldsymbol{E}=-j\omega\mu\boldsymbol{H}, \tag{IV}$$

where $\boldsymbol{E}$[V/m] is the electric field, $\boldsymbol{H}$[A/m] the magnetic field in the medium; μ[H/m] is the permeability, ϵ[F/m] the permittivity and σ [S/m] the conductivity of the medium.

Taking the divergence of (III) and substituting from (I) thereto, a generalized relation of continuity is obtained as

$$\nabla_Q\cdot\boldsymbol{J}+\left(j\omega+\frac{\sigma}{\epsilon}\right)\rho_e=0. \tag{1.1}$$

From (II), $\boldsymbol{H}$ can be represented as

$$\boldsymbol{H}=\frac{1}{\mu}\nabla_Q\times\boldsymbol{A} \tag{1.2}$$

due to a theorem in the vector analysis[1]. $\boldsymbol{A}$ is called the vector potential and capable of being given an infinite number of functions.

Substituting from (1.2) into (IV)

$$\nabla_Q\times(\boldsymbol{E}+j\omega\boldsymbol{A})=0$$

is obtained. Applying a theorem in the vector analysis[2], the last equation will lead to

$$\boldsymbol{E}+j\omega\boldsymbol{A}=-\nabla_Q\phi,$$

where ϕ is the scalar potential. The electric field is then represented by the equation

$$\boldsymbol{E}=-\nabla_Q\phi-j\omega\boldsymbol{A}. \tag{1.3}$$

1) I. Murakami, 'Mathematics for electromagnetic theory', Vol. I, 1976, Hirokawa Publ. Co. Ltd., pp. 69–70, theorem 3.

2) ditto. pp. 67–68, theorems 1 and 2.

Substituting from (1.2) and (1.3) into (III) and using (V. 19)

$$\nabla_Q \nabla_Q \cdot \boldsymbol{A} - \Delta_Q \boldsymbol{A} - k^2 \boldsymbol{A} + (j\omega\mu\epsilon + \mu\sigma) \nabla_Q \phi = \mu \boldsymbol{J}. \tag{1.4}$$

Substituting from (1.3) into (I)

$$-\nabla_Q \cdot \nabla_Q \phi - j\omega \nabla_Q \cdot \boldsymbol{A} = \frac{\rho_e}{\epsilon}. \tag{1.5}$$

Since, as has been stated above, $\boldsymbol{A}$ can be chosen from among an infinite number of functions, the following supplementary condition may be imposed

$$\nabla_Q \cdot \boldsymbol{A} + (j\omega\mu\epsilon + \mu\sigma) \phi = 0. \tag{1.6}$$

(1.6) may be called a generalized Lorentz auxiliary condition. Upon substitution of this equation into (1.4) and (1.5), these two equations are transformed to

$$\Delta_Q A + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}, \tag{1.7}$$

$$\nabla_Q{}^2 \phi + k^2 \phi = -\frac{\rho_e}{\epsilon}, \tag{1.8}$$

respectively. (1.8) is the inhomogeneous Helmholtz differential equation which defines the scalar potential ϕ due to the electric charge ρ_e, and (1.7) the inhomogeneous vector Helmholtz differential equation which defines the vector potential $\boldsymbol{A}$ due to the electric current $\boldsymbol{J}$, under the condition set out in (1.6).

1.2 Case of magnetic sources

Consider, in this case, the electromagnetic field due to magnetic sources, which are characterized by a continuous distribution in the medium of the magnetic charge ρ_m [Wb/m^3] and the magnetic current $\boldsymbol{J}_m$ [V/m^2] instead of ρ_e and $\boldsymbol{J}$ considered in the foregoing section.

Maxwell's equations may then be written as[1)]

$$\nabla_Q \cdot \boldsymbol{E} = 0, \tag{I'}$$

$$\nabla_Q \cdot \boldsymbol{H} = \frac{\rho_m}{\mu}, \tag{II'}$$

$$\nabla_Q \times \boldsymbol{H} = (j\omega\epsilon + \sigma) \boldsymbol{E}, \tag{III'}$$

$$\nabla_Q \times \boldsymbol{E} = -\boldsymbol{J}_m - j\omega\mu \boldsymbol{H}. \tag{IV'}$$

From (I′) $\boldsymbol{E}$ can be represented as

$$\boldsymbol{E} = -\frac{1}{\epsilon} \nabla_Q \times \boldsymbol{A}_m \tag{1.9}$$

due to the same reasoning as before. $\boldsymbol{A}_m$ is called the magnetic vector potential and may exist infinite.

Substituting from (1.9) into (III′)

1) I. Murakami, 'Mathematics for electromagnetic theory', II, (1976, Hirokawa) p. 274

$$\nabla_Q \times \left\{ \boldsymbol{H} + \left(j\omega + \frac{\sigma}{\epsilon} \right) \boldsymbol{A}_m \right\} = 0 \tag{1.10}$$

is obtained. From a similar reasoning to before are derived the following relations

$$\boldsymbol{H} + \left(j\omega + \frac{\sigma}{\epsilon} \right) \boldsymbol{A}_m = -\nabla_Q \phi_m$$

or

$$\boldsymbol{H} = -\nabla_Q \phi_m - \left(j\omega + \frac{\sigma}{\epsilon} \right) \boldsymbol{A}_m , \tag{1.11}$$

where ϕ_m is called the magnetic scalar potential.

Substituting from (1.9) and (1.11) into (II′) and (IV′)

$$\nabla_Q \cdot \nabla_Q \phi_m + \left(j\omega + \frac{\sigma}{\epsilon} \right) \nabla_Q \cdot \boldsymbol{A}_m = -\frac{\rho_m}{\mu} , \tag{1.12}$$

$$-\nabla_Q \nabla_Q \cdot \boldsymbol{A}_m + \Delta_Q \boldsymbol{A}_m = -\epsilon \boldsymbol{J}_m + j\omega\mu\epsilon \nabla_Q \phi_m - k^2 \boldsymbol{A}_m \tag{1.13}$$

are obtained. It is possible to impose the condition due to the reason mentioned above,

$$\nabla_Q \cdot \boldsymbol{A}_m + j\omega\mu\epsilon\phi_m = 0 , \tag{1.14}$$

which is called the Lorentz auxiliary condition in the magnetic case. Upon subsitution from (1.14) into (1.12) and (1.13), these two equations are transformed to

$$\nabla_Q{}^2 \phi_m + k^2 \phi_m = -\frac{\rho_m}{\mu} , \tag{1.15}$$

$$\Delta_Q \boldsymbol{A}_m + k^2 \boldsymbol{A}_m = -\epsilon \boldsymbol{J}_m , \tag{1.16}$$

respectively. (1.15) is the inhomogeneous Helmholtz differential equation which defines the magnetic potential ϕ_m due to the magnetic charge ρ_m, and (1.16) the inhomogeneous vector Helmholtz differential equation which defines the magnetic vector potential $\boldsymbol{A}_m$ due to the magnetic current $\boldsymbol{J}_m$, under the condition set out in (1.14).

2 Solution in an integral form of inhomogeneous Helmholtz differential equation

2.1 Derivation of Green's theorems

Consider a domain V enclosed by a surface S in an infinite, homogeneous, isotropic and conducting medium characterized by $(\mu, \epsilon, \sigma; k)$ as shown in Fig. 1, where Q (ξ, η, ζ) denotes a running source point and P (x, y, z) the point of observation. Assume that the functions ϕ and ψ are continuously differentiable scalar functions throughout the medium, and construct a vector therefrom as

$$\boldsymbol{A} = \psi \nabla_Q \phi . \tag{2.1}$$

Substitute from (2.1) into the divergence theorem (V. 25):

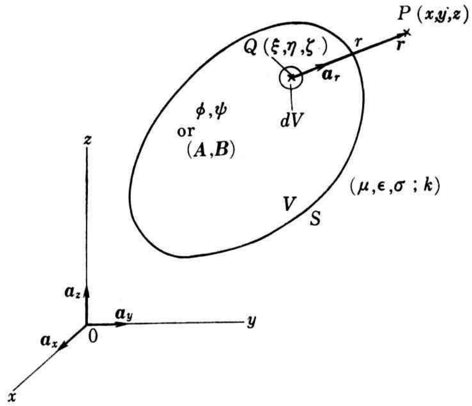

Fig. 1 For Green's theorems

$$\int_V \nabla_Q \cdot \boldsymbol{A}\, dV = \int_S \boldsymbol{A} \cdot \boldsymbol{a}_n\, dS\,. \tag{2.2}$$

From (V. 13)

$$\nabla_Q \cdot \boldsymbol{A} = \nabla_Q \cdot (\psi \nabla_Q \phi) = \nabla_Q \psi \cdot \nabla_Q \phi + \psi \nabla_Q^2 \phi\,.$$

Substituting from this equation into (2.2)

$$\int_V (\nabla_Q \psi \cdot \nabla_Q \phi + \psi \nabla_Q^2 \phi)\, dV = \int_S \psi \nabla_Q \phi \cdot \boldsymbol{a}_n\, dS \tag{2.3}$$

is obtained. (2.3) is called the Green's first theorem. Interchanging the roles of ϕ and ψ in this equation and subtracting the ensuing equation from (2.3),

$$\int_V (\psi \nabla_Q^2 \phi - \phi \nabla_Q^2 \psi)\, dV = \int_S (\psi \nabla_Q \phi - \phi \nabla_Q \psi) \cdot \boldsymbol{a}_n\, dS \tag{2.4}$$

is obtained, since $\nabla_Q \psi \cdot \nabla_Q \phi = \nabla_Q \phi \cdot \nabla_Q \psi$. (2.4) is called the Green's second theorem.

2.2 $\psi = \dfrac{e^{-jkr}}{r}$ is a solution to the homogeneous Helmholtz equations $\nabla_Q^2 \psi + k^2 \psi = 0$ and $\nabla_P^2 \psi + k^2 \psi = 0$

As shown in Fig. 2, $\boldsymbol{r}$ is the distance vector oriented from a source point Q (ξ, η, ζ) to the point of observation P (x, y, z) and represented as

$$\boldsymbol{r} = \boldsymbol{a}_x (x-\xi) + \boldsymbol{a}_y (y-\eta) + \boldsymbol{a}_z (z-\zeta) = \boldsymbol{a}_r r\,,$$

$$r = \sqrt{(x-\xi)^2 + (y-\xi)^2 + (z-\zeta)^2}\,,$$

r being the distance and $\boldsymbol{a}_r$ the unit vector oriented along the direction from Q to P.

Consider now a scalar spherical wave function

$$\psi = \frac{e^{-jkr}}{r}\,, \tag{2.5}$$

where $k = \sqrt{\omega^2 \mu \epsilon - j\omega\mu\sigma}$ [rad/m] is the propagation constant. For this function

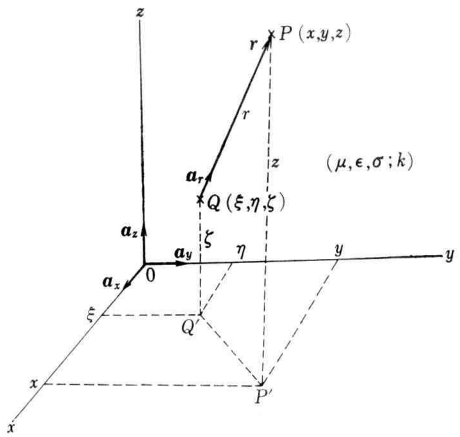

Fig. 2 Distance vector: $\boldsymbol{r}=\boldsymbol{a}_x(x-\xi)+\boldsymbol{a}_y(y-\eta)+\boldsymbol{G}_2(z-\zeta)=\boldsymbol{a}_r r$, $r=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$

$$
\begin{aligned}
\nabla_Q\psi &= \boldsymbol{a}_x\frac{\partial\psi}{\partial\xi}+\boldsymbol{a}_y\frac{\partial\psi}{\partial\eta}+\boldsymbol{a}_z\frac{\partial\psi}{\partial\zeta}=\boldsymbol{a}_x\frac{-jkr-1}{r^2}\frac{\partial r}{\partial\xi}e^{-jkr}+\cdots\\
&=\boldsymbol{a}_x\frac{-jkr-1}{r^2}\frac{-(x-\xi)}{r}e^{-jkr}+\cdots=\boldsymbol{a}_x\frac{(x-\xi)}{r}\left(jk+\frac{1}{r}\right)\psi+\cdots\\
&=\boldsymbol{a}_r\left(jk+\frac{1}{r}\right)\psi. \qquad (2.6)
\end{aligned}
$$

Likewise, the following equation is obtained

$$\nabla_P\psi=-\boldsymbol{a}_r\left(jk+\frac{1}{r}\right)\psi=-\nabla_Q\psi. \qquad (2.6')$$

Using the above result

$$
\begin{aligned}
\nabla_Q^2\psi &= \nabla_Q\cdot\nabla_Q\psi=\frac{\partial}{\partial\xi}\left\{\frac{x-\xi}{r}\left(jk+\frac{1}{r}\right)\frac{e^{-jkr}}{r}\right\}+\cdots\\
&=\left\{\frac{-r-\frac{\partial r}{\partial\xi}(x-\xi)}{r^2}\left(jk+\frac{1}{r}\right)\frac{e^{-jkr}}{r}+\frac{x-\xi}{r}\frac{-\frac{\partial r}{\partial\xi}}{r}\frac{e^{-jkr}}{r}\right.\\
&\quad\left.+\left(\frac{x-\xi}{r}\right)^2\left(jk+\frac{1}{r}\right)^2\frac{e^{-jkr}}{r}\right\}+\cdots\\
&=\left(-\frac{3jk}{r}-\frac{3}{r^2}+\frac{jk}{r}+\frac{1}{r^2}+\frac{1}{r^2}-k^2+\frac{2jk}{r}+\frac{1}{r^2}\right)\frac{e^{-jkr}}{r}=-k^2\psi.
\end{aligned}
$$

As a consequence, ψ is verified to satisfy the homogeneous Helmholtz differential equation:

$$\nabla_Q^2\psi+k^2\psi=0. \qquad (2.7)$$

Likewise, the following equaltion is verified:

$$\nabla_P^2\psi+k^2\psi=0\,. \tag{2.7'}$$

2.3 Solution in an integral from of inhomogeneous differential equation: $\nabla_Q^2\phi+k^2\phi=-\frac{\rho_e}{\epsilon}$

Consider a distribution of electric charge ρ_e[C/m³] in space which is characterized by a continuously differentiable function. The potential function ϕ for this charge distribution then satisfies the following inhomogeneous Helmholtz equation throughout the region where ρ_e is distributed

$$\nabla_Q^2\phi+k^2\phi=-\frac{\rho_e}{\epsilon}\,, \tag{2.8}$$

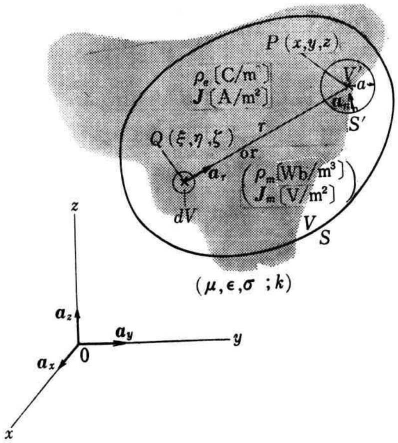

Fig. 3 For the solutions in integral forms of inhomogencous Helmholtz and vector Helmholtz differential equations

Assume, as shown in Fig. 3, that a point of observation P is located in V and that a sufficiently small sphere S' enclosing V' centered on P and with radius a is completely included within V. Applying the Green's second theorem (2.4) to the domain $V-V'$ enclosed by $S+S'$ and using ψ given by (2.5) and ϕ as the electric potential ϕ

$$\int_{V-V'}(\psi\nabla_Q^2\phi-\phi\nabla_Q^2\psi)\,dV=\int_{S+S'}(\psi\nabla_Q\phi-\phi\nabla_Q\psi)\cdot\boldsymbol{a}_n\,dS \tag{2.9}$$

is obtained. The surface integral over S' appearing on the right hand side of this equation is transformed as follows

$$\int_{S'}(\psi\nabla_Q\phi-\phi\nabla_Q\psi)\cdot\boldsymbol{a}_n\,dS=\int_{S'}\left\{\frac{e^{-jka}}{a}\frac{\partial\phi}{\partial n}-\phi\left(jk+\frac{1}{a}\right)\frac{e^{-jka}}{a}\right\}dS\,.$$

where the relations $r=a$ and $\boldsymbol{a}_n=\boldsymbol{a}_r$ have been used. Denoting by $\overline{\frac{\partial\phi}{\partial n}}$ the mean value of $\frac{\partial\phi}{\partial n}$ within V' and by $\bar{\phi}$ that of ϕ, the right hand side of the equation is further transformed to

$$= \left\{ \frac{e^{-jka}}{a} \frac{\overline{\partial\phi}}{\partial n} - \bar{\phi}\left(jk + \frac{1}{a}\right) \frac{e^{-jka}}{a} \right\} 4\pi a^2 .$$

Assuming that a tends to zero, this equation is reduced to

$$= -4\pi\phi(P), \tag{2.10}$$

where $\phi(P)$ denotes the potential at P. Since, in this case, $V-V'\rightarrow V$, substitution from (2.7), (2.8) and (2.10) into (2.9) leads to

$$\phi(P) = \frac{1}{4\pi}\int_V \frac{\rho_e}{\epsilon}\psi dV + \frac{1}{4\pi}\int_S (\psi \nabla_Q \phi - \phi \nabla_Q \psi) \cdot \boldsymbol{a}_n dS . \tag{2.11}$$

This is a solution in an integral form of the inhomogeneous Helmholtz differentil equation (2.8).

2.4 Solution in an integral form of inhomogeneous vector Helmholtz differential equation: $\Delta_Q \boldsymbol{A} + k^2\boldsymbol{A} = -\mu \boldsymbol{J}$

Consider a distribution of electric current $\boldsymbol{J}$[A/m^2] in space which is characterized by a continuously differentiable vector function. The vector potential function $\boldsymbol{A}$ [Wb/m] should satisfy the following vector Helmholtz differential equation

$$\Delta_Q \boldsymbol{A} + k^2\boldsymbol{A} = -\mu \boldsymbol{J} . \tag{2.12}$$

as has been stated in Section 1.1; the electric charge density ρ_e and the current density $\boldsymbol{J}$ have to satisfy the continuity condition (1.1).

Henceforth, the mathematical manipulations will be made with reference to a rectangular coordinates (x, y, z). (2.12) is then written as

$$\boldsymbol{a}_x(\nabla_Q^2 A_x + k^2 A_x) + \boldsymbol{a}_y(\nabla_Q^2 A_y + k^2 A_y) + \boldsymbol{a}_z(\nabla_Q^2 A_z + k^2 A_z) = -\mu(\boldsymbol{a}_x J_x + \boldsymbol{a}_y J_y + \boldsymbol{a}_z J_z)$$

or

$$\begin{aligned} \nabla_Q^2 A_x + k^2 A_x &= -\mu J_x , \\ \nabla_Q^2 A_y + k^2 A_y &= -\mu J_y , \\ \nabla_Q^2 A_z + k^2 A_z &= -\mu J_z . \end{aligned} \tag{2.13}$$

The individual equation in (2.13) is an inhomogeneous Helmholtz differential equation for each component of $\boldsymbol{A} = \boldsymbol{a}_x A_x + \boldsymbol{a}_y A_y + \boldsymbol{a}_z A_z$ and the integral solutions for them are given in similar expressions to (2.11) as follows

$$\begin{aligned} A_x(P) &= \frac{1}{4\pi}\int_V \mu J_x \psi dV + \frac{1}{4\pi}\int_S (\psi \nabla_Q A_x - A_x \nabla_Q \psi) \cdot \boldsymbol{a}_n dS , \\ A_y(P) &= \frac{1}{4\pi}\int_V \mu J_y \psi dV + \frac{1}{4\pi}\int_S (\psi \nabla_Q A_y - A_y \nabla_Q \psi) \cdot \boldsymbol{a}_n dS , \\ A_x(P) &= \frac{1}{4\pi}\int_V \mu J_z \psi dV + \frac{1}{4\pi}\int_S (\psi \nabla_Q A_z - A_z \nabla_Q \psi) \cdot \boldsymbol{a}_n dS . \end{aligned} \tag{2.14}$$

Multiplying the both sides of each equation in (2.14) by $\boldsymbol{a}_x$, $\boldsymbol{a}_y$ and $\boldsymbol{a}_z$, and summing up the three equations, $\boldsymbol{A}(P)$ is obtained as

$$A\,(P) = \frac{1}{4\pi}\int_V \mu J\psi dV + \frac{1}{4\pi}\int_S \{\psi(a_n\cdot\nabla_Q)A - Aa_n\cdot\nabla_Q\psi\}dS\,, \tag{2.15}$$

where

$$(a_n\cdot\nabla_Q)A = a_x(a_n\cdot\nabla_Q A_x) + a_y(a_n\cdot\nabla_Q A_y) + a_z(a_n\cdot\nabla_Q A_z)\,,$$

the last equation being defined in a vector analysis[1)].

(2.15) may also be expressed as

$$A\,(P) = \frac{1}{4\pi}\int_V \mu J\psi dV + \frac{1}{4\pi}\int_S \left(\psi\frac{\partial A}{\partial n} - A\frac{\partial\psi}{\partial n}\right)dS\,, \tag{2.15'}$$

where

$$\frac{\partial A}{\partial n} = (a_n\cdot\nabla_Q)A\,, \qquad \frac{\partial\psi}{\partial n} = a_n\cdot\nabla_Q\psi\,.$$

A solution in an integral form of an inhomogeneous vector Helmholtz differential equation can directly be derived using the vector (analogue of) Creen's second theorem. This method will be explained in the succeeding two sections. The identity of the result from the method and (2.15) is verified in Section 5.

3 Solution in an integral form of inhomogeneous vector Helmholtz differential equation

3.1 Vector (analogue of) Green's theorem

Consider, in Fig. 1, two vectors A and B which are continuously differentiable vector quantities defined in space.

Construct a vector

$$B\times\nabla_Q\times A \tag{3.1}$$

and substitute from this vector into A appearing in divergence theorem (2.2),

$$\int_V \nabla_Q\cdot(B\times\nabla_Q\times A)dV = \int_S B\times\nabla_Q\times A\cdot a_n\,dS$$

is then obtained. Since, from (V. 17),

$$\nabla_Q\cdot(B\times\nabla_Q\times A) = \nabla_Q\times A\cdot\nabla_Q\times B - B\cdot\nabla_Q\times\nabla_Q\times A\,,$$

the last tow equations lead to

$$\int_V(\nabla_Q\times A\cdot\nabla_Q\times B - B\cdot\nabla_Q\times\nabla_Q\times A)dV = \int_S(B\times\nabla_Q\times A)\cdot a_n\,dS\,. \tag{3.2}$$

This is called the vector (analogue of) Green's first theorem.

Making a similar equation to (3.2), but, in which the roles of A and B are interchanged; substracting (3.2) from the equation just mentioned and using

$$\nabla_Q\times B\cdot\nabla_Q\times A = \nabla_Q\times A\cdot\nabla_Q\times B\,,$$

the following equation is derived

1) I. Murakami, 'Mathematics for electromagnetic theory,' I, (1976, Hirokawa), p. 47

$$\int_V (\boldsymbol{B}\cdot\nabla_Q\times\nabla_Q\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla_Q\times\nabla_Q\times\boldsymbol{B})\,dV=\int_S (\boldsymbol{A}\times\nabla_Q\times\boldsymbol{B}-\boldsymbol{B}\times\nabla_Q\times\boldsymbol{A})\cdot\boldsymbol{a}_n\,dS\,. \tag{3.3}$$

This is called the vector (analogue of) Green's second theorem.

3.2 Solution in an integral form of inhomogeneous vector Helmholtz differential equation: $\Delta_Q \boldsymbol{A}+k^2\boldsymbol{A}=-\mu\boldsymbol{J}$

In the following, (3.3) is applied to the domain $V-V'$ which is enclosed by $S+S'$ as shown in Fig. 3.

Now, assume that the vector potential $\boldsymbol{A}$ is equated to $\boldsymbol{A}$ appearing in (3.3) and $\psi\boldsymbol{a}$ to $\boldsymbol{B}$, $\boldsymbol{a}$ being a constant unit vector with magnitude unity and being oriented along an arbitrary fixed direction throughout an infinite space.

First, the individual vector terms appearing under the integral sign with respect to V in (3.3) are transformed as follows.

(V. 19) and (1.7) lead to

$$(\nabla_Q\times\nabla_Q\times\boldsymbol{A})=\nabla_Q\nabla_Q\cdot\boldsymbol{A}-\Delta_Q\boldsymbol{A}=\nabla_Q\nabla_Q\cdot\boldsymbol{A}+k^2\boldsymbol{A}+\mu\boldsymbol{J}\,, \tag{3.5}$$

and applciation of (V. 15) leads to

$$(\nabla_Q\times\boldsymbol{B})=\nabla_Q(\psi\boldsymbol{a})=\nabla_Q\psi\times\boldsymbol{a}+\psi\nabla_Q\times\boldsymbol{a}=\nabla_Q\psi\times\boldsymbol{a}\,, \tag{3.6}$$

because a differentiation of a constant unit vector $\boldsymbol{a}$ gives a result of null.

Applciation of (V. 18) to rotation of (3.6) leads to

$$\begin{aligned}(\nabla_Q\times\nabla_Q\times\boldsymbol{B})&=\nabla_Q\times(\nabla_Q\psi\times\boldsymbol{a})=\nabla_Q\psi\nabla_Q\cdot\boldsymbol{a}-\boldsymbol{a}\nabla_Q\cdot\nabla_Q\psi+(\boldsymbol{a}\cdot\nabla_Q)\nabla_Q\psi-(\nabla_Q\psi\cdot\nabla_Q)\boldsymbol{a}\\&=-\boldsymbol{a}\nabla_Q^2\psi+(\boldsymbol{a}\cdot\nabla_Q)\nabla_Q\psi=\boldsymbol{a}k^2\psi+(\boldsymbol{a}\cdot\nabla_Q)\nabla_Q\psi\,,\end{aligned}$$

since $\nabla_Q\cdot\boldsymbol{a}$ and $(\nabla_Q\psi\cdot\nabla_Q)\,\boldsymbol{a}$ are zero due to a similar reasoning to before and by the use of (2.7).

In addition, from (V. 16)

$$\nabla_Q(\boldsymbol{a}\cdot\nabla_Q\psi)=(\boldsymbol{a}\cdot\nabla_Q)\nabla_Q\psi+(\nabla_Q\psi\cdot\nabla_Q)\boldsymbol{a}+\boldsymbol{a}\times\nabla_Q\times\nabla_Q\psi+\nabla_Q\psi\times\nabla_Q\times\boldsymbol{a}=(\boldsymbol{a}\cdot\nabla_Q)\nabla_Q\psi\,,$$

since $(\nabla_Q\psi\cdot\nabla_Q)\,\boldsymbol{a}$ and $\nabla_Q\times\boldsymbol{a}$ are zero due to a similar reasoning to before and by the use of $\nabla_Q\times\nabla_Q\psi\boldsymbol{a}$ which itself is due to (V. 21). As a consequence

$$(\nabla_Q\times\nabla_Q\times\boldsymbol{B})=\boldsymbol{a}k^2\psi+\nabla_Q(\boldsymbol{a}\cdot\nabla_Q\psi)\,. \tag{3.7}$$

Secondly, the individual vector terms appearing under the integral sign with respect to S in (3.3) are transformed as follows.

Applciation of (V. 2) leads to

$$(\boldsymbol{A}\times\nabla_Q\times\boldsymbol{B})=\boldsymbol{A}\times(\nabla_Q\psi\times\boldsymbol{a})=(A\cdot\boldsymbol{a})\nabla_Q\psi-(\boldsymbol{A}\cdot\nabla_Q\psi)\boldsymbol{a} \tag{3.8}$$

and

$$(\boldsymbol{B}\times\nabla_Q\times\boldsymbol{A})=\psi\boldsymbol{a}\times\nabla_Q\times\boldsymbol{A} \tag{3.9}$$

Substitution from (3.5), (3.7) through (3.9) into the bracketted terms appearing under the integral sign with respect to V in (3.3) yields

$$\begin{aligned}&(\boldsymbol{B}\cdot\nabla_Q\times\nabla_Q\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla_Q\times\nabla_Q\times\boldsymbol{B})\\&=\psi\boldsymbol{a}\cdot(\nabla_Q\nabla_Q\cdot\boldsymbol{A})+k^2\psi\boldsymbol{a}\cdot\boldsymbol{A}+\mu\boldsymbol{a}\cdot\boldsymbol{J}\psi-\boldsymbol{A}\cdot\boldsymbol{a}k^2\psi-\boldsymbol{A}\cdot\nabla_Q(\boldsymbol{a}\cdot\nabla_Q\psi)\,.\end{aligned}$$

Applying (V. 13) twice

$$\begin{aligned}\nabla_Q\cdot(\psi\boldsymbol{a}\nabla_Q\cdot\boldsymbol{A}) &= (\nabla_Q\nabla_Q\cdot\boldsymbol{A})\cdot\boldsymbol{a}\psi+\nabla_Q\cdot\boldsymbol{A}\nabla_Q\cdot(\psi\boldsymbol{a})\\ &= (\nabla_Q\nabla_Q\cdot\boldsymbol{A})\cdot\boldsymbol{a}\psi+\nabla_Q\cdot\boldsymbol{A}(\nabla_Q\psi\cdot\boldsymbol{a}),\end{aligned}$$

since $\nabla_Q\cdot\boldsymbol{a}=0$,

$$(\boldsymbol{B}\cdot\nabla_Q\times\nabla_Q\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla_Q\times\nabla_Q\times\boldsymbol{B})=\nabla_Q\cdot(\psi\boldsymbol{a}\nabla_Q\cdot\boldsymbol{A})-(\boldsymbol{a}\cdot\nabla_Q\psi)\nabla_Q\cdot\boldsymbol{A}-\boldsymbol{A}\cdot\nabla_Q(\boldsymbol{a}\cdot\nabla_Q\psi)+\mu\boldsymbol{a}\cdot\boldsymbol{J}\psi.$$

From (V. 13)

$$\nabla_Q\cdot\{(\boldsymbol{a}\cdot\nabla_Q\psi)\boldsymbol{A}\}=\nabla_Q(\boldsymbol{a}\cdot\nabla_Q\psi)\cdot\boldsymbol{A}+(\boldsymbol{a}\cdot\nabla_Q\psi)\nabla_Q\cdot\boldsymbol{A}.$$

The volume integral in (3.3) is then transformed to

$$\int_{V-V'}[\nabla_Q\cdot(\psi\boldsymbol{a}\nabla_Q\cdot\boldsymbol{A})-\nabla_Q\cdot\{(\boldsymbol{a}\cdot\nabla_Q\psi)\boldsymbol{A}\}+\mu\boldsymbol{a}\cdot\boldsymbol{J}\psi]dV.$$

Application of the divergence theorem (V. 25) to the first two integrals appearing in the last equation leads to

$$\begin{aligned}&=\int_{S+S'}\{\psi\boldsymbol{a}\nabla_Q\cdot\boldsymbol{A}-(\boldsymbol{a}\cdot\nabla_Q\psi)\boldsymbol{A}\}\cdot\boldsymbol{a}_n\,dS+\int_{V-V'}\mu\boldsymbol{a}\cdot\boldsymbol{J}\psi dV\\ &=\boldsymbol{a}\cdot\int_{S+S'}\{\psi\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}-(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi\}\,dS+\boldsymbol{a}\cdot\int_{V-V'}\mu\boldsymbol{J}\psi dV.\end{aligned}$$

Using (3.6), the bracketted terms appearing under the integral sign with respect to $S+S'$ in (3.3) yields

$$\begin{aligned}(\boldsymbol{A}\times\nabla_Q\times\boldsymbol{B}-\boldsymbol{B}\times\nabla_Q\times\boldsymbol{A}) &= \boldsymbol{A}\times(\nabla_Q\psi\times\boldsymbol{a})-\psi\boldsymbol{a}\times\nabla_Q\times\boldsymbol{A}\\ &= (\boldsymbol{A}\cdot\boldsymbol{a})\nabla_Q\psi-(\boldsymbol{A}\cdot\nabla_Q\psi)\boldsymbol{a}-\psi\boldsymbol{a}\times\nabla_Q\times\boldsymbol{A}\end{aligned}$$

from (V. 2).

The surface integral of (3.3) is then transformed to

$$\begin{aligned}&\int_{S+S'}\{(\boldsymbol{A}\cdot\boldsymbol{a})\nabla_Q\psi-(\boldsymbol{A}\cdot\nabla_Q\psi)\boldsymbol{a}-\psi\boldsymbol{a}\times\nabla_Q\times\boldsymbol{A}\}\cdot\boldsymbol{a}_n\,dS\\ &\quad=\int_{S+S'}\{(\boldsymbol{A}\cdot\boldsymbol{a})(\nabla_Q\psi\cdot\boldsymbol{a}_n)-(\boldsymbol{A}\cdot\nabla_Q\psi)(\boldsymbol{a}\cdot\boldsymbol{a}_n)-\psi\boldsymbol{a}_n\cdot(\boldsymbol{a}\times\nabla_Q\times\boldsymbol{A})\}dS.\end{aligned}$$

Applying (V.1) to the third term in the above integrand, the last surface integral is reduced to

$$=\boldsymbol{a}\cdot\int_{S+S'}\{(\boldsymbol{a}_n\cdot\nabla_Q\psi)\boldsymbol{A}-\boldsymbol{a}_n(\boldsymbol{A}\cdot\nabla_Q\psi)+\psi\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\}dS.$$

Applying from (V.2) the relation

$$(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi=-(\nabla_Q\psi\cdot\boldsymbol{A})\boldsymbol{a}_n+(\nabla_Q\psi\cdot\boldsymbol{a})\boldsymbol{A},$$

the surface integral is further reduced to

$$=\boldsymbol{a}\cdot\int_{S+S'}\{(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi+\psi\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\}dS.$$

(3.3) is therefore written as follows

$$\boldsymbol{a}\cdot\int_{V-V'}\mu\boldsymbol{J}\psi\,dV=\boldsymbol{a}\cdot\int_{S+S'}\{-\psi\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}+(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi+(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi+\psi\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\}dS\,.$$

Since $\boldsymbol{a}$ is a constant unit vector oriented along an arbitrary direction, an identity between the integrals themselves should be satisfied, namely

$$\int_{V-V'}\mu\boldsymbol{J}\psi\,dV=\int_{S+S'}\{-\psi\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}+(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi+(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi+\psi\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\}dS\,. \tag{3.10}$$

Consider then the surface integral over S' where $r=a$ and $\boldsymbol{a}_n=\boldsymbol{a}_r$:

$$\int_{S'}\Big\{-\frac{e^{-jka}}{a}\boldsymbol{a}_r\nabla_Q\cdot\boldsymbol{A}+(\boldsymbol{a}_r\cdot\boldsymbol{A})\boldsymbol{a}_r\Big(jk+\frac{1}{a}\Big)\frac{e^{-jka}}{a}+(\boldsymbol{a}_r\times\boldsymbol{A})\times\boldsymbol{a}_r\Big(jk+\frac{1}{a}\Big)\frac{e^{-jka}}{a}+\frac{e^{-jka}}{a}\boldsymbol{a}_r\times\nabla_Q\times\boldsymbol{A}\Big\}dS\,.$$

Applciation of (V.5) to the second and the thrid terms appearing in the above integrand yields

$$=\int_{S'}\Big\{-\frac{e^{-jka}}{a}\boldsymbol{a}_r\nabla_Q\cdot\boldsymbol{A}+\boldsymbol{A}\Big(jk+\frac{1}{a}\Big)\frac{e^{-jka}}{a}+\frac{e^{-jka}}{a}\boldsymbol{a}_r\times\nabla_Q\times\boldsymbol{A}\Big\}dS\,. \tag{3.11}$$

If a is assumed to tend to zero, the last formula is equated to

$$=4\pi\boldsymbol{A}(P)\,.$$

Summarizing, the solution $\boldsymbol{A}(P)$ is derived as follows:

$$\boldsymbol{A}(P)=\frac{1}{4\pi}\int_V\mu\boldsymbol{J}\psi\,dV+\frac{1}{4\pi}\int_S\{\psi\boldsymbol{a}_n\nabla_Q\cdot A-(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi-(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi-\psi\boldsymbol{a}_n\times\nabla_Q\times A\}dS\,. \tag{3.12}$$

This is the solution in an integral form of (3.4)

4 Solution in an integral form of inhomogeneous scalar and vector Helmholtz differential equations for magnetic case

Assume that the magnetic charge ρ_m [Wb/m^3] and the magnetic current $\boldsymbol{J}_m$ [V/m^2] which are characterized by a continuously differeniable function respectively are distributed in an infinite, homogeneous, isotropic and conducting medium. In this case, the confinuity condition

$$\nabla_Q\cdot\boldsymbol{J}_m+j\omega\rho_m=0 \tag{4.1}$$

should be satisfied throughout the medium.

4.1 Solution in an integral form of inhomogeneous Helmoltz differential equation:

$$\nabla_Q^2\phi_m+k^2\phi_m=-\frac{\rho_m}{\mu}$$

The solution can be derived in a similar fashion to Section 2, and the result is given as follows:

$$\phi_m(P) = \frac{1}{4\pi}\int_V \frac{\rho_m}{\mu}\psi\, dV + \frac{1}{4\pi}\int_S (\psi \nabla_Q \phi_m - \phi_m \nabla_Q \psi)\cdot \boldsymbol{a}_n\, dS\,. \tag{4.2}$$

4.2 Solution in an integral form of inhomogeneous vector Helmholtz differential equation $\Delta_Q \boldsymbol{A}_m + k^2 \boldsymbol{A}_m = -\epsilon \boldsymbol{J}_m$

The solution is derived in a similar fashion to Section 2, and the results is given as follows:

$$\boldsymbol{A}_m(P) = \frac{1}{4\pi}\int_V \epsilon \boldsymbol{J}_m \psi\, dV + \frac{1}{4\pi}\int_S \{\psi\, \boldsymbol{a}_n \nabla_Q \cdot \boldsymbol{A}_m - (\boldsymbol{a}_n \cdot \boldsymbol{A}_m)\nabla_Q \psi - (\boldsymbol{a}_n \times \boldsymbol{A}_m)\times \nabla_Q \psi - \psi \boldsymbol{a}_n \times \nabla_Q \times \boldsymbol{A}_m\} dS \tag{4.3}$$

and

$$\boldsymbol{A}_m(P) = \frac{1}{4\pi}\int_V \epsilon \boldsymbol{J}_m \psi\, dV + \frac{1}{4\pi}\int_S \{\psi (\boldsymbol{a}_n \cdot \nabla_Q)\boldsymbol{A}_m - \boldsymbol{A}_m \boldsymbol{a}_n \cdot \nabla_Q \psi\}\, dS \tag{4.4}$$

or

$$\boldsymbol{A}_m(P) = \frac{1}{4\pi}\int_V \epsilon \boldsymbol{J}_m \psi\, dV + \frac{1}{4\pi}\int_S \left(\psi \frac{\partial \boldsymbol{A}_m}{\partial n} - \boldsymbol{A}_m \frac{\partial \psi}{\partial n}\right) dS \tag{4.4'}$$

5 Identity of (2.17) and (3.11); (4.3) and (4.4)

In the foregoing sections have been obtained $\boldsymbol{A}(P)$ and $\boldsymbol{A}_m(P)$ in two different expressions. However, the identity of (2.17) and (3.11) is verified as follows.

$$\begin{aligned}
\{\boldsymbol{a}_n \times \nabla_Q) \times (\psi \boldsymbol{A})\}_\xi &= (\boldsymbol{a}_n \times \nabla_Q)_\eta (\psi A_\zeta) - (\boldsymbol{a}_n \times \nabla_Q)_\zeta (\psi A_\eta) \\
&= \left(\cos\gamma \frac{\partial \psi}{\partial \xi} - \cos\alpha \frac{\partial \psi}{\partial \zeta}\right) A_\zeta + \psi \left(\cos\gamma \frac{\partial A_\zeta}{\partial \xi} - \cos\alpha \frac{\partial A_\zeta}{\partial \zeta}\right) \\
&\quad - \left(\cos\alpha \frac{\partial \psi}{\partial \eta} - \cos\beta \frac{\partial \psi}{\partial \xi}\right) A_\eta - \psi \left(\cos\alpha \frac{\partial A_\eta}{\partial \eta} - \cos\beta \frac{\partial A_\eta}{\partial \xi}\right) \\
&= (\boldsymbol{a}_n \times \nabla_Q \psi)_\eta A_\zeta - (\boldsymbol{a}_n \times \nabla_Q \psi) A_\eta \\
&\quad + \psi \left(\cos\gamma \frac{\partial A_\zeta}{\partial \xi} - \cos\alpha \frac{\partial A_\zeta}{\partial \zeta} - \cos\alpha \frac{\partial A_\eta}{\partial \eta} + \cos\beta \frac{\partial A_\eta}{\partial \xi}\right) \\
&= \{(\boldsymbol{a}_n \times \nabla_Q \psi) \times \boldsymbol{F}\}_\xi + \psi \left(\cos\alpha \frac{\partial A_\xi}{\partial \xi} + \cos\beta \frac{\partial A_\eta}{\partial \xi} + \cos\gamma \frac{\partial A_\zeta}{\partial \xi}\right) \\
&\quad - \psi \left(\cos\alpha \frac{\partial A_\xi}{\partial \xi} + \cos\alpha \frac{\partial A_\eta}{\partial \eta} + \cos\alpha \frac{\partial A_\zeta}{\partial \zeta}\right) \\
&= \{(\boldsymbol{a}_n \times \nabla_Q \psi) \times \boldsymbol{A}\}_\xi - \psi \cos\alpha\, \nabla_Q \cdot \boldsymbol{A} + \psi F_\xi
\end{aligned}$$

The η and ζ components of $\{(\boldsymbol{a}_r \times \nabla_Q) \times (\psi \boldsymbol{A})\}$ are obtained likewise, resulting in

$$(\boldsymbol{a}_n \times \nabla_Q) \times (\psi \boldsymbol{A}) = (\boldsymbol{a}_n \times \nabla_Q \psi) \times \boldsymbol{A} - \psi\, \boldsymbol{a}_n \nabla_Q \cdot \boldsymbol{A} + \psi \boldsymbol{F}\,. \tag{5.1}$$

On the other hand

$$(\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A})_\xi=\cos\beta\Big(\frac{\partial A_\eta}{\partial\xi}-\frac{\partial A_\xi}{\partial\eta}\Big)-\cos\gamma\Big(\frac{\partial A_\xi}{\partial\zeta}-\frac{\partial A_\zeta}{\partial\xi}\Big)$$

$$=\Big(\cos\alpha\frac{\partial A_\xi}{\partial\xi}+\cos\beta\frac{\partial A_\eta}{\partial\xi}+\cos\gamma\frac{\partial A_\zeta}{\partial\xi}\Big)$$

$$-\Big(\cos\alpha\frac{\partial A_\xi}{\partial\xi}+\cos\beta\frac{\partial A_\xi}{\partial\eta}+\cos\gamma\frac{\partial A_\xi}{\partial\zeta}\Big)$$

$$=F_\xi-\boldsymbol{a}_n\cdot\nabla_Q A_\xi\,,$$

then

$$\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\,\psi=\psi\,\boldsymbol{F}-(\boldsymbol{a}_n\cdot\nabla_Q)\,\boldsymbol{A} \tag{5.2}$$

is obtained. Substituting from (5.2) into (5.1)

$$\begin{aligned}(\boldsymbol{a}_n\times\nabla_Q)\times(\psi\boldsymbol{A}) &= (\boldsymbol{a}_n\times\nabla_Q)\times\boldsymbol{A}-\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}+\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\,\psi+(\boldsymbol{a}_n\cdot\nabla_Q)\,\boldsymbol{A}\,\psi\\ &= (\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi-(\boldsymbol{a}_n\nabla_Q\psi)\,\boldsymbol{A}+(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi\\ &\quad-\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}+\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\,\psi+(\boldsymbol{a}_n\cdot\nabla_Q)\,\boldsymbol{A}\,\psi\,,\end{aligned}$$

and using the relations

$$\begin{aligned}(\boldsymbol{a}_n\times\nabla_Q\psi)\times\boldsymbol{A} &= -\,(\boldsymbol{A}\cdot\nabla_Q\psi)\,\boldsymbol{a}_n+(\boldsymbol{A}\cdot\boldsymbol{a}_n)\nabla_Q\psi\,,\\ (\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi &= -\,(\boldsymbol{A}\cdot\nabla_Q\psi)\,\boldsymbol{a}_n+(\boldsymbol{a}_n\cdot\nabla_Q\psi)\,\boldsymbol{A}\,,\end{aligned}$$

finally

$$\begin{aligned}(\boldsymbol{a}_n\times\nabla_Q)\times(\psi\boldsymbol{A}) &= (\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi+(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi+\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\psi-\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}\\ &\quad+\psi\,(\boldsymbol{a}_n\cdot\nabla_Q)\,\boldsymbol{A}-\boldsymbol{A}\,\nabla_Q\psi\cdot\boldsymbol{a}_n\end{aligned} \tag{5.3}$$

is obtained.

From (V. 1)

$$\boldsymbol{a}\cdot(\boldsymbol{a}_n\times\nabla_Q)\times(\psi\boldsymbol{A})=(\boldsymbol{a}_n\times\nabla_Q)\cdot(\psi\boldsymbol{A}\cdot\boldsymbol{a})=\boldsymbol{a}_n\cdot\nabla_Q\times(\psi\boldsymbol{A}\times\boldsymbol{a})\,,$$

then, referring to (V. 28)

$$\boldsymbol{a}\cdot\int_S(\boldsymbol{a}_n\times\nabla_Q)\times(\psi\boldsymbol{A})\,dS=\int_S\nabla_Q\times(\psi\boldsymbol{A}\times\boldsymbol{a})\cdot d\boldsymbol{S}=\int_C(\psi\boldsymbol{A}\times\boldsymbol{a})\cdot d\boldsymbol{c}=0$$

is obtained. Since, in the last equation, S is a closed surface, C does not exist and the intgral should therefore vanish. As a consequence

$$\boldsymbol{a}\cdot\int_S\{\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}-(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi-(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi-\psi\,\boldsymbol{a}_n\times\boldsymbol{A}\}dS$$

$$=\boldsymbol{a}\cdot\int_S\{\psi\,(\boldsymbol{a}_n\cdot\nabla_Q)\,\boldsymbol{A}-\boldsymbol{A}\,\nabla_Q\psi\cdot\boldsymbol{a}_n\}dS\,,$$

which leads to

$$\int_S\{\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}-(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi-(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi-\psi\,\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\}dS$$

$$=\int_S \{\psi(\boldsymbol{a}_n\cdot\nabla_Q)\boldsymbol{A}-\boldsymbol{A}\nabla_Q\psi\cdot\boldsymbol{a}_n\}dS.$$

Identity of (2.17) and (3.11) has thus been verified. Identity of (4.3) and (4.4) can also be verified quite likewise.

6 Radiation formulae

6.1 Electric case

Let the distribution of ρ_e and $\boldsymbol{J}$ be restricted to remain within the domain V, and let the surface S be removed with V extending to infinity in all directions (Fig. 4). Provided that k is purely real, the surface integral will remain finite. However, in a medium met in reality, k is possessed of a very small imaginary part even in a perfect dielectric medium. The surface integral will therefore vanish due to the attenuation, and the following equations should be satisfied.

$$\phi(P)=\frac{1}{4\pi}\int_V \frac{\rho_e}{\epsilon}\psi\,dV, \tag{6.1}$$

$$\boldsymbol{A}(P)=\frac{1}{4\pi}\int_V \mu\boldsymbol{J}\psi\,dV. \tag{6.2}$$

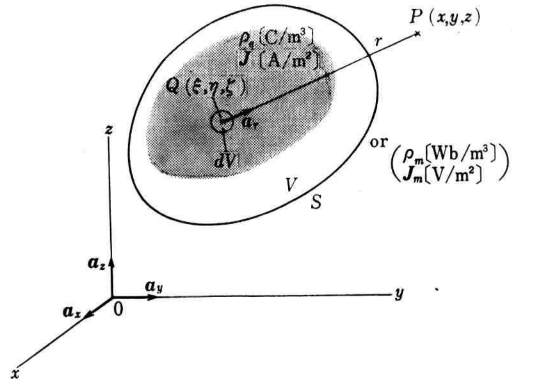

Fig. 4 For radiation formula (Sources exist in V only.)

6.2 Magentic case

Let the distribution of ρ_m and $\boldsymbol{J}_m$ be restricted to remain within the domain V, and let the surface S be removed with V extending to infinity in all directions. Then, by a similar reasoning to the foregoing case are derived the following equations.

$$\phi_m(P)=\frac{1}{4\pi}\int_V \frac{\rho_m}{\mu}\psi\,dV, \tag{6.3}$$

$$\boldsymbol{A}_m(P)=\frac{1}{4\pi}\int_V \epsilon\boldsymbol{J}_m\psi\,dV. \tag{6.4}$$

7 Huygens principle for potential functions

7.1 Electric case

When the electric charge ρ_e and current $\boldsymbol{J}$ are distributed outside of S but not in the domain V (Fig. 5), the potential functions due to the soruces and at an arbitrary point P in the domain V are given by

$$\phi(P)=\frac{1}{4\pi}\int_S(\psi\nabla_Q\phi-\phi\nabla_Q\psi)\cdot\boldsymbol{a}_n\,dS\,, \tag{7.1}$$

$$A(P)=\frac{1}{4\pi}\int_S\{\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}-(\boldsymbol{a}_n\cdot\boldsymbol{A})\nabla_Q\psi-(\boldsymbol{a}_n\times\boldsymbol{A})\times\nabla_Q\psi-\psi\,\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}\}dS\,. \tag{7.2}$$

(7.1) and (7.2) may be called representing the Huygens principle in terms of the scalar potential ϕ and the vector potential $\boldsymbol{A}$ respectively.

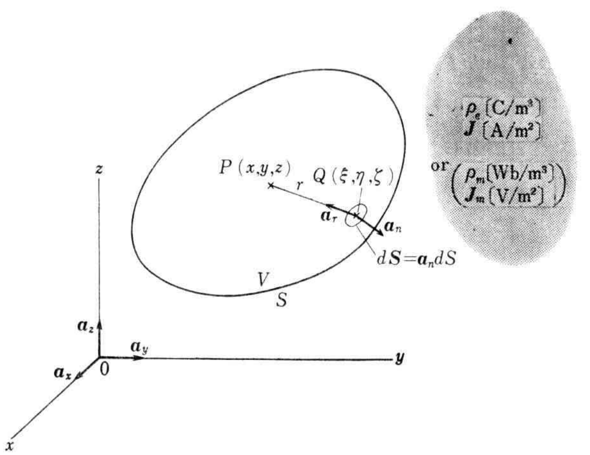

Fig. 5 For Huygens principle (Sources exist out of S only.)

7.2 Magnetic case

When the magnetic charge ρ_m and magnetic current $\boldsymbol{J}_m$ are distributed outside of S, but not in the domain V, the potential functions due to the soruces and at an arbitrary point P located in the domain V are given by

$$\phi_m(P)=\frac{1}{4\pi}\int_S(\psi\nabla_Q\,\phi_m-\phi_m\nabla_Q\psi)\cdot\boldsymbol{a}_n dS\,, \tag{7.3}$$

$$\begin{aligned}A_m(P)=\frac{1}{4\pi}\int_S&\{\psi\,\boldsymbol{a}_n\nabla_Q\cdot\boldsymbol{A}_m-(\boldsymbol{a}_n\cdot\boldsymbol{A}_m)\nabla_Q\psi\\&-(\boldsymbol{a}_n\times\boldsymbol{A}_m)\times\nabla_Q\psi-\psi\,\boldsymbol{a}_n\times\nabla_Q\times\boldsymbol{A}_m\}dS\,.\end{aligned} \tag{7.4}$$

(7.3) and (7.4) may be called representing the Huygens principle in terms of the scalar potential ϕ_m and the vector potential $\boldsymbol{A}_m$ respectively.

8. Conclusion

The solutions in an integral form of inhomogeneous Helmhotz and vector Helmholtz differential equations have been derived for the electric case and for the magnetic case, and the duality of the two solutions, namely $\boldsymbol{A}(P)$ and $\boldsymbol{A}_m(P)$ derived in different ways respectively, have been verified.

It is essential to recognize the fact that the principle of duality holds between the corresponding electric and magnetic equations.

In a future paper continued from this, the Hyugens principle in terms of the electric field and the magnetic field will be discussed.

Appendix Formulae of vector analysis.

(V. 1) $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})$: circulation law of scalar triple product

(V. 2) $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$: vector triple product

$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})+\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A})+\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B})=0$

(V. 3) $(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$

(V. 4) $(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{D})\boldsymbol{C}-(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{D}$

(V. 5) $\boldsymbol{B} = (\boldsymbol{a}\cdot\boldsymbol{B})\boldsymbol{a}+(\boldsymbol{a}\times\boldsymbol{B})\times\boldsymbol{a}$

(V. 6) $\dfrac{d}{dt}(\boldsymbol{A}+\boldsymbol{B}) = \dfrac{d\boldsymbol{A}}{dt}+\dfrac{d\boldsymbol{B}}{dt}$

(V. 7) $\dfrac{d}{dt}(\phi\boldsymbol{A}) = \dfrac{d\phi}{dt}\boldsymbol{A}+\phi\dfrac{d\boldsymbol{A}}{dt}$

(V. 8) $\dfrac{d}{dt}(\boldsymbol{A}\cdot\boldsymbol{B}) = \dfrac{d\boldsymbol{A}}{dt}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\dfrac{d\boldsymbol{B}}{dt}$

(V. 9) $\dfrac{d}{dt}(\boldsymbol{A}\times\boldsymbol{B}) = \dfrac{d\boldsymbol{A}}{dt}\times\boldsymbol{B}+\boldsymbol{A}\times\dfrac{d\boldsymbol{B}}{dt}$

(V 10) $\nabla(\phi+\psi) = \nabla\phi+\nabla\psi$

(V. 11) $\nabla(\phi\psi) = \psi\nabla\phi+\phi\nabla\psi$

(V. 12) $\nabla\cdot(\boldsymbol{A}+\boldsymbol{B}) = \nabla\cdot\boldsymbol{A}+\nabla\cdot\boldsymbol{B}$

(V. 13) $\nabla\cdot(\phi\boldsymbol{A}) = \nabla\psi\cdot\boldsymbol{A}+\phi\nabla\cdot\boldsymbol{A}$

(V. 14) $\nabla\times(\boldsymbol{A}+\boldsymbol{B}) = \nabla\times\boldsymbol{A}+\nabla\times\boldsymbol{B}$

(V. 15) $\nabla\times(\phi\boldsymbol{A}) = \nabla\phi\times\boldsymbol{A}+\phi\nabla\times\boldsymbol{A}$

(V 16) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\times\nabla\times\boldsymbol{B}+\boldsymbol{B}\times\nabla\times\boldsymbol{A}$

(V. 17) $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\boldsymbol{B}$

(V. 18) $\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}\nabla\cdot\boldsymbol{B}-\boldsymbol{B}\nabla\cdot\boldsymbol{A}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$

(V. 19) $\nabla\times\nabla\times\boldsymbol{A} = \nabla\nabla\cdot\boldsymbol{A}-\Delta\boldsymbol{A}$

(V. 20) $(\nabla\times\boldsymbol{A})\times\boldsymbol{A}=\frac{1}{2}\nabla A^2+(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}$

(V. 21) $\nabla\times\nabla\phi=0$

(V. 22) $\nabla\cdot\nabla\times\boldsymbol{A}=0$

Distance vector $\boldsymbol{r}$ from a point $Q(\xi,\eta,\zeta)$ to another point $P(x,y,z)$ is given by $\boldsymbol{r}=\boldsymbol{a}_x(x-\xi)+\boldsymbol{a}_y(y-\eta)+\boldsymbol{a}_z(z-\zeta)=\boldsymbol{a}_r r$, $r=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$, where $\boldsymbol{a}_x, \boldsymbol{a}_y, \boldsymbol{a}_z$ are fundamental vectors in the cartesian coordinates (x,y,z), $\boldsymbol{a}_r$ the unit vector from $Q$to$P$, r the distance between Q and P.

(V. 23) $\nabla_P\cdot\boldsymbol{r}=-\nabla_Q\cdot\boldsymbol{r}=3, \quad \nabla_P\times\boldsymbol{r}=\nabla_Q\times\boldsymbol{r}=0,$

$$\nabla_P\left(\frac{1}{r}\right)=-\nabla_Q\left(\frac{1}{r}\right)=-\boldsymbol{a}_r\frac{1}{r^2}, \quad \nabla_P^2\left(\frac{1}{r}\right)=\nabla_Q^2\left(\frac{1}{r}\right)=0$$

Concerning any surface S surrounded by a closed path C, or any domain V enclosed by a closed surface S.

(V. 24) $\int_V \nabla\phi\, dV=\int_S \phi\, d\boldsymbol{S}$

(V. 25) $\int_V \nabla\cdot\boldsymbol{A}\, dV=\int_S \boldsymbol{A}\cdot d\boldsymbol{S}$: divergence theorem

(V. 26) $\int_V \nabla\times\boldsymbol{A}\, dV=\int_S \boldsymbol{A}\times d\boldsymbol{S}$

(V. 27) $\int_S d\boldsymbol{S}\times\nabla\phi=\int_C \phi\, d\boldsymbol{c}$

(V. 28) $\int_S \nabla\times\boldsymbol{A}\cdot d\boldsymbol{S}=\int_C \boldsymbol{A}\cdot d\boldsymbol{c}$: Stokes' theorem

(V. 29) $\int_V (\nabla\psi\cdot\nabla\phi+\psi\nabla^2\phi)\, dV=\int_S \psi\nabla\phi\cdot d\boldsymbol{S}$: Green's 1st theorem

(V. 30) $\int_V (\psi\nabla^2\phi-\phi\nabla^2\psi)\, dV=\int_S (\psi\nabla\phi-\phi\nabla\psi)\cdot d\boldsymbol{S}$: Green's 2nd theorem

(V. 31) $\int_V (\nabla\times\cdot\boldsymbol{A}\nabla\times\boldsymbol{B}-\boldsymbol{A}\cdot\nabla\times\nabla\times\boldsymbol{B})\, dV=\int_S \boldsymbol{A}\times\nabla\times\boldsymbol{B}\cdot d\boldsymbol{S}$: vector (analogue of) Green's 1st theorem

(V. 32) $\int_V (\boldsymbol{B}\cdot\nabla\times\nabla\times\boldsymbol{A}-\boldsymbol{A}\cdot\nabla\times\nabla\times\boldsymbol{B})\, dV=\int_S (\boldsymbol{A}\times\nabla\times\boldsymbol{B}-\boldsymbol{B}\times\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S}$

: vector (anatogue of) Green's 2nd theorem