

On Curves with Weierstrass Points whose First Non-gaps are Six

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Abstract

Let C be a (complete non-singular) curve over an algebraically closed field of characteristic 0. For any point P of C , $H(P)$ denotes the set of non-gaps n at P , i.e., non-negative integers n such that $H^0(C, \mathcal{O}_c((n-1)P)) \subset H^0(C, \mathcal{O}_c(nP))$. For any positive integer a we denote by W_a^2 the set of linearly equivalent classes of divisors D on C of degree a with $\dim H^0(C, \mathcal{O}_c(D)) \geq 2$. First we show the necessary and sufficient condition on C with $W_6^2 = \phi$ and $W_6^2 \neq \phi$ such that $\#W_6^2 \geq 2$ and we describe the set W_6^2 . Next we consider the case where C is a curve of genus ≥ 26 with a point P such that $H(P)$ starts with 6. Then we rewrite the above condition in terms of $H(P)$. Lastly we give examples of points on curves whose first non-gaps are six.

Introduction

Let \mathbf{N} be the additive semigroup of non-negative integers. A subsemigroup H of \mathbf{N} is called a *numerical semigroup* if the complement $\mathbf{N}-H$ of H in \mathbf{N} is finite. The number of the set $\mathbf{N}-H$ is called the *genus* of H , which is denoted by $g(H)$. Let k be an algebraically closed field of characteristic 0 and let C be a complete non-singular 1-dimensional algebraic variety over k (which is called a *curve*) of genus g . For any point P of C , a non-negative integer n is called a *gap* at P if

$$H^0(C, \mathcal{O}_c((n-1)P)) = H^0(C, \mathcal{O}_c(nP)).$$

Then the set $H(P)$ of non-gaps at P becomes a numerical semigroup of genus g . For a numerical semigroup H , if a curve C has a point P satisfying $H(P) = H$, then we say that C *admits* H . Moreover, if a is the least positive integer which belongs to H , then we say that H *starts with* a . If $H(P)$ starts with a , then by abuse of language a is called the *first non-gap* of P . Let C be a curve of genus $g \geq 1$ and $J(C)$ the Jacobian variety of C . For any positive integer a , we define a map $j_a: S^a C \rightarrow J(C)$ by sending an element D of $S^a C$ to the divisor class of $D - aP_0$ where $S^a C$ and P_0 denote the a -th symmetric product of C and a fixed point of C respectively. For any positive integer m , we set

$$W_a^m(C) = \{j_a(D) | l(D) \geq m\}$$

where we set $l(D) = \dim H^0(C, \mathcal{O}_c(D))$.

Now we are in the following situation: let a be a fixed positive integer and let C be a curve of genus g with a point P whose first non-gap is a . If a is prime and if g is sufficiently large, then $W_a^2(C)$ consists of only one point¹⁾. On the other hand in the case $a=4$ and $g \geq$

10 we have $\#W_4^2(C) \geq 2$ if and only if $H(P)$ contains 6^{11} . In this paper we are concerned with the case $a=6$. In § 2 we will show the main result as follows.

Main Theorem. *For $g \geq 26$, we have $\#W_6^2 \geq 2$ if and only if one of the following cases occurs :*

- (1) $H(P) \ni 8$ and $H(P) \ni 10$,
- (2) $H(P) \ni 9$,
- (3) $H(P) \ni 8$ and $H(P) \ni 10$,
- (4) $H(P) \ni 8$, $H(P) \ni 10$ and $H(P) \ni 14$,
- (5) $H(P) \ni 8$, $H(P) \ni 10$, $H(P) \ni 14$ and $H(P) \ni 16$.

Moreover, in the case (1) $\dim W_6^2=2$, in the case (2) $\dim W_6^2=1$ and W_6^2 is irreducible, in the cases (3) and (4) $\dim W_6^2=1$, and in the case (5) W_6^2 consists of two points.

In § 3 we will discuss the existence of curves with points whose first non-gaps are six. In the case $a \leq 4$, for any numerical semigroup H starting with a there exists a curve C which admits $H^{(12)}$. In this section we will calculate the semigroup H of non-gaps of a fixed point of an automorphism T on a curve C of order 6 satisfying $C/\langle T \rangle \cong \mathbf{P}^1$ where $\langle T \rangle$ denotes the group generated by T (such a H is said to be 6-cyclic) and using this result we will give examples of the Main Theorem. Moreover, it is showed that there are examples of non-6-cyclic numerical semigroups starting with 6 which curves admit.

§ 1. On 6-gonal Curves with $\#W_6^2 \geq 2$.

First we give the estimate of the genus of a curve with certain conditions and deduce the useful result from the estimate.

Theorem 1.1. *Let C be a curve of genus g and $K(C)$ the function field of C . Let K_1 and K_2 be subfields of $K(C)$ with their compositum $K(C)$. If n_i is the degree $[K(C): K_i]$ of $K(C)$ over K_i and g_i is the genus of K_i for $i=1, 2$, then we have*

$$g \leq (n_1-1)(n_2-1) + n_1g_1 + n_2g_2.$$

This theorem is due to Castelnuovo. For its proof see e.g. 1) in Reference.

Corollary 1.2. *Let C be a curve of genus g , let p be a prime number, and let K_0 be a subfield of $K(C)$ of genus g' with $[K(C): K_0] = p$.*

- (1) *If x is a function on C such that the degree of x , i.e., the degree of the polar divisor of x , is less than $(g-pg')/(p-1)+1$, then x lies in K_0 and p divides the degree of x .*
- (2) *If $g > (p-1)^2 + 2pg'$, then K_0 is a unique subfield of $K(C)$ of genus g' with $[K(C): K_0] = p$.*

For this proof, see e.g. 1) in Reference.

Let C be a curve of genus $g \geq 1$ and let $j_n: S^n C \rightarrow J(C)$ be the morphism defined by sending D to the divisor class of $D - nP_0$, with a fixed point P_0 of C . We denote by $W_n = W_n(C)$ the image of j_n and by $W_n^m = W_n^m(C)$ the set $\{j_n(D) | l(D) \geq m\}$. Moreover, \bar{W}_n^m denotes the complement of $W_{n-1}^m + W_1$ in W_n^m . For $n \geq 2$, C is said to be n -gonal if we have $W_{n-1}^2 = \emptyset$ and $W_n^2 \neq \emptyset$. Then we see:

Proposition 1.3. *Let C be an n -gonal curve of genus $g \geq (n-1)^2 + 1$. If for any*

integer m with $2 \leq m \leq n-1$ which divides n and for any integer r with $1 \leq r \leq (n-m)^2/m^2$, C is not an m -sheeted covering of a curve of genus r , then W_n^2 consists of only one point.

For any divisors D and D' on C , $D \sim D'$ means that D and D' are linearly equivalent. For any element x of $K(C)$ we denote by $(x)_\infty$ the polar divisor of x . If two elements x, y of $K(C)-k$ satisfy $k(x)=k(y)$, then we say that x and y are equivalent. Then we get the following:

Remark 1.4. (1) If two elements x and y of $K(C)-k$ are equivalent, then we have $(x)_\infty \sim (y)_\infty$.

(2) For any positive integer n , we have a bijection between the set of equivalence classes of elements x of $K(C)-k$ of degree n with $l((x)_\infty)=2$ and the set W_n^2 by sending the equivalence class of x to $j_n((x)_\infty)$.

Proposition 1.5. Let n be an integer with $n \geq 3$ and let p be a prime number with $2 \leq p \leq n-1$ and $p|n$. Let C be an n -gonal curve of genus g and let C' be an n/p -gonal curve of genus g' . Suppose that there exists a finite morphism $f: C \rightarrow C'$ of degree p . If $n < (g - pg')/(p-1) + 1$, then we get a bijective morphism between $W_{n/p}^2(C')$ and $W_n^2(C)$ by sending the divisor class of $D' - (n/p)f(P_0)$ to that of $f^*D' - nP_0$ where f^*D' denotes the inverse image of the divisor D' by f .

Proof. Let $J(f): J(C') \rightarrow J(C)$ be the map defined by sending the divisor class of D' to that of f^*D' . First we show that $J(f)(W_{n/p}^2(C')) = W_n^2(C)$. In the proof, for any divisor D we denote by $cl(D)$ the divisor class of D . For any $x' \in K(C')-k$ we have

$$J(f)(cl((x')_\infty - (n/p)f(P_0))) = cl((f^*x')_\infty - nP_0)$$

and

$$\deg(f^*x')_\infty = [K(C):K(C')][K(C'):k(x')] = p \deg x',$$

where $f^*: K(C') \rightarrow K(C)$ is the inclusion corresponding to the morphism $f: C \rightarrow C'$. Using Remark 1.4 (2) we get $J(f)(W_{n/p}^2(C')) \subseteq W_n^2(C)$. Conversely, let $x \in K(C)$ of degree n with $l((x)_\infty)=2$. Applying Corollary 1.2 (2) to the case $K_0 = K(C')$ we get $x' \in K(C')$ with $f^*x' = x$. Hence we see

$$n = \deg x = [K(C):K(C')][K(C'):k(x')] = p \deg x',$$

which implies that $cl((x')_\infty) \in W_{n/p}^2(C')$. From the above it follows that $W_n^2(C) \subseteq J(f)(W_{n/p}^2(C'))$. Hence we obtain $J(f)(W_{n/p}^2(C')) = W_n^2(C)$. Let x' and y' be elements of $K(C')$ of degree n/p such that

$$l((x')_\infty) = l((y')_\infty) = 2 \text{ and } j_n(f^*((x')_\infty)) = j_n(f^*((y')_\infty)).$$

It follows from Remark 1.4 (2) that $f^*(x')$ and $f^*(y')$ are equivalent, i.e., $k(x') = k(y')$. Hence by Remark 1.4 (1) we get $cl((x')_\infty) = cl((y')_\infty)$. It follows from the above that $J(f)$ induces a bijection between $W_{n/p}^2(C')$ and W_n^2 . Q.E.D.

Applying Proposition 1.3 to the case $n=6$ we get the following:

Remark 1.6. If C is a 6-gonal curve of genus $g \geq 26$ with $\#W_6^2(C) \geq 2$, then it is either a 3-sheeted covering of an elliptic curve or a double covering of a non-hyperelliptic curve of genus 3 or a double covering of a non-hyperelliptic curve of genus 4.

Proposition 1.7. Let C be a curve of genus g .

(1) If $g \geq 14$ and if C is a 3-sheeted covering of an elliptic curve, then C is 6-gonal, and W_6^2 is 1-dimensional and irreducible.

(2) If $g \geq 12$ and if C is a double covering of a non-hyperelliptic curve of genus 3, then C is 6-gonal and W_6^2 is 1-dimensional.

(3) If $g \geq 14$ and if C is a double covering of a non-hyperelliptic curve of genus 4 with no half canonical divisor D' (resp. with a half canonical divisor D') with $l(D')=2$, then C is 6-gonal and W_6^2 consists of two points (resp. one point).

Proof. (1) Let $f: C \rightarrow E$ be a 3-sheeted covering of an elliptic curve E . Applying Corollary 1.2 (2) to the case $p=3$, $g'=1$ and $K_0=K(E)$, we get $W_5^2=\phi$. Since E is elliptic, there exists $x' \in K(E)$ of degree 2. Hence $\deg f^*(x')=6$, which implies that $W_6^2 \neq \phi$, i.e., C is 6-gonal. Using Proposition 1.5 in the case $n=6$, $p=3$, $g \geq 14$ and $g'=1$ we have a bijective morphism between $W_2^2(E)$ and W_6^2 . Hence $\dim W_6^2=1$ and W_6^2 is irreducible.

(2) Let $f: C \rightarrow C'$ be a double covering of a non-hyperelliptic curve C' of genus 3. Then by Corollary 1.2 (2) we have $W_5^2=\phi$. Since C' is trigonal, C is 6-gonal. Applying Proposition 1.5 to this case we get a bijective morphism between $W_3^2(C')$ and W_6^2 . In view of $\dim W_3^2(C')=1$ we get $\dim W_6^2=1$.

(3) Let $f: C \rightarrow C'$ be a double covering of a non-hyperelliptic curve C' of genus 4. In the same way as in the proof of (2), we see that C is 6-gonal and that there is a bijective morphism between $W_3^2(C')$ and W_6^2 . If C' has no half canonical divisor D' (resp. has a half canonical divisor D') with $l(D')=2$, then $W_3^2(C')$ consists of two points (resp. one point).³⁾ Hence we get our desired result. Q.E.D.

Combining Proposition 1.7 with Remark 1.6 we obtain the following :

Theorem 1.8. *Let C be a 6-gonal curve of genus $g \geq 26$. Then the following are equivalent :*

- (a) $\# W_6^2 \geq 2$,
- (b) C is either a 3-sheeted covering of an elliptic curve or a double covering of a non-hyperelliptic curve of genus 3 or a double covering of a non-hyperelliptic curve of genus 4 with no half canonical divisor D' with $l(D')=2$.

§ 2. On Curves with Points whose First Non-gaps are Six.

In this section we are always in the following situation : let C be a curve of genus g with a point P such that $H(P)$ starts with 6, and let x be a function on C such that $(x)_\infty = 6P$. Moreover, for positive integers a_1, \dots, a_n , $\langle a_1, \dots, a_n \rangle$ denotes the additive semigroup generated by a_1, \dots, a_n . First we describe conditions on C and $H(P)$ such that C is 6-gonal.

Lemma 2.1. *Let $g \geq 21$. Then C is 6-gonal if and only if C is not a double covering of a curve of genus 2.*

Proof. Suppose that C is a double covering of a curve of genus 2. Then $W_4^2 \neq \phi$, which implies that C is not 6-gonal. This proves the "only if" part. Suppose that C is not a double covering of a curve of genus 2. Since $j_6((x)_\infty) \in W_6^2$, it suffices to show that $W_5^2 = \phi$.

Suppose that $W_5^2 \neq \phi$. Then one of the following cases occurs: 1) $W_2^2 = \phi$ and $W_3^2 \neq \phi$, 2) $W_3^2 = \phi$ and $W_4^2 \neq \phi$, 3) $W_4^2 = \phi$ and $W_5^2 \neq \phi$. In the case 1) we have $W_3^2 = W_3^2 \neq \phi$, which implies that there is a function y on C of degree 3. If $K(C) = k(x, y)$, then by Theorem 1.1. we get

$$g \leq (\deg x - 1)(\deg y - 1) = 10.$$

This is a contradiction to $g \geq 21$. Hence we have inclusions

$$K(C) \supset k(x, y) = k(y) \supset k(x).$$

Since $k(x, y)$ is rational and $[K(C) : k(x, y)] = 3$, $H(P)$ must contain 3. This is a contradiction. In the case 2) there exists a function y on C of degree 4. Then we have $[K(C) : k(x, y)] = 2$. Let C' be the curve corresponding to $k(x, y)$. Since the function x (resp. y) on C' is of degree 3 (resp. 2), by Theorem 1.1 the genus g' of C' is less than or equal to 2. If $g' = 0$, then C is hyperelliptic, which is a contradiction. If $g' = 1$, we have $H(P) \ni 4$. This is a contradiction. If $g' = 2$, C is a double covering of the curve C' of genus 2. This is a contradiction. In the case 3), using Theorem 1.1 we must have $g \leq 20$. This is a contradiction. Hence we get $W_5^2 = \phi$. Q.E.D.

Lemma 2.2. (1) If $g \geq 10$ and if C is a double covering of a curve of genus 2, then $H(P) \ni 8, 10$.

(2) If $g \geq 13$ and $H(P) \ni 8, 10$, then C is a double covering of a curve of genus 2. In this case, we have $\dim W_6^2 = 2$.

Proof. (1) Let $f : C \rightarrow C'$ be a double covering of a curve C' of genus 2. Applying Corollary 1.2 (1) to the case $p=2$, $g'=2$, $g \geq 10$ and $K_0 = K(C')$, we get inclusions $k(x) \subset K(C') \subset K(C)$. Since $H(f(P))$ is either $\langle 2, 5 \rangle$ or $\langle 3, 4, 5 \rangle$, $H(P)$ must contain 8 and 10.

(2) From $g \geq 13$ and $H(P) \ni 8, 10$ it follows that $H(P) \ni 7, 9, 11, 13, 15, 17$. For example, if $17 \in H(P)$, then we have

$$g \leq \#(N - \langle 6, 8, 10, 17 \rangle) = 12.$$

This is a contradiction. Hence we obtain

$$H(P) = \{0, 6, 8, 10, 12, 14, 16, 18, 19, \dots\},$$

which implies that C is a double covering of a curve of genus 2.⁴⁾ Hence C is a 4-gonal curve. Suppose that C is a double covering of an elliptic curve. Then using Corollary 1.2 (1) we have $H(P) \ni 4$. This is a contradiction. Since

$$\dim W_6^2 > \dim W_5^2 > \dim W_4^2 \geq 0,$$

we have $\dim W_6^2 \geq 2$. Since C is a 4-gonal curve which is not a double covering of an elliptic curve, we must have $\dim W_6^2 = 2$.⁵⁾

By Lemmas 2.1 and 2.2 we get

Proposition 2.3. If $g \geq 21$, then the following are equivalent :

- (a) C is 6-gonal,
- (b) C is not a double covering of a curve of genus 2,
- (c) $H(P) \ni 8$ or $H(P) \ni 10$.

The remainder of this section is devoted to describing Theorem 1.8 in terms of $H(P)$.

Lemma 2.4. (1) If $g \geq 15$ and if C is a 3-sheeted covering of an elliptic curve, then $H(P) \ni 9$.

(2) If $g \geq 20$ and $H(P) \ni 9$, then C is a 3-sheeted covering of an elliptic curve.

Proof. (1) Let $f: C \rightarrow E$ be a 3-sheeted covering of an elliptic curve E . Applying Corollary 1.2 (1) to the case $p=3$, $g'=1$, $g \geq 15$ and $K_0 = K(E)$, we get inclusions $k(x) \subset K(E) \subset K(C)$. Since $H(f(P))$ is $\langle 2, 3 \rangle$, we must have $H(P) \ni 9$.

(2) From $g \geq 20$ and $H(P) \ni 9$, it follows that

$$H(P) = \{0, 6, 9, 12, 15, 18, 19, \dots\}.$$

Let y be a function such that $(y)_\infty = 9P$. Then we have a relation

$$y^2 = Q_1(x)y + Q_2(x)$$

where $Q_i(x) \in k[x]$ with $i=1, 2$, $\deg Q_1(x) \leq 1$ and $\deg Q_2(x) = 3$. If we denote by C' the curve corresponding to $k(x, y)$, then C is a 3-sheeted covering of C' . Moreover, the genus g' of C' is less than or equal to 1. Hence $g' = 1$. Q.E.D.

Lemma 2.5. (1) If $g \geq 16$ and if C is a double covering of a curve of genus 3, one of the following cases occurs: (a) $H(P) \ni 8$, $H(P) \ni 10$, (b) $H(P) \ni 8$, $H(P) \ni 10, 14$.

(2) If $g \geq 18$ and if $H(P) \ni 8$, $H(P) \ni 10$ (resp. $H(P) \ni 8$, $H(P) \ni 10, 14$), then C is a double covering of a non-hyperelliptic curve of genus 3.

Proof. (1) Let $f: C \rightarrow C'$ be a double covering of a curve C' of genus 3. Since $k(x) \subset K(C')$, we have $H(f(P)) \ni 3$, which implies that $H(f(P))$ is either $\langle 3, 4 \rangle$ or $\langle 3, 5, 7 \rangle$. If $H(f(P)) = \langle 3, 4 \rangle$ (resp. $\langle 3, 5, 7 \rangle$), then the case (a) (resp. (b)) occurs.

(2) First, consider the case of $H(P) \ni 8$, $H(P) \ni 10$. The assumption $g \geq 18$ implies that

$$H(P) = \{0, 6, 8, 12, 14, 16, 18, 20, 22, 24, 25, \dots\}$$

If y is a function with $(y)_\infty = 8P$, then we have a relation

$$y^3 = Q_1(x)y^2 + Q_2(x)y + Q_3(x)$$

where $Q_i(x) \in k[x]$, $\deg Q_1(x) \leq 1$, $\deg Q_2(x) \leq 2$ and $\deg Q_3(x) = 4$. Let C' be the curve corresponding to $k(x, y)$. Then the genus g' of C' is less than or equal to $(4-1)(4-2)/2 = 3$ and C is a double covering of C' . From $H(P) \ni 4, 10$ it follows that C' is a non-hyperelliptic curve of genus 3. Next, suppose that $H(P) \ni 8$ and $H(P) \ni 10, 14$. Then we have

$$H(P) = \{0, 6, 10, 12, 14, 16, 18, 20, 22, 24, 25, \dots\}.$$

It y and z are functions such that $(y)_\infty = 10P$ and $(z)_\infty = 14P$, then we have two relations

$$y^2 = Q_1(x)z + Q_2(x)y + Q_3(x)$$

and

$$yz = Q_4(x)z + Q_5(x)y + Q_6(x)$$

where $Q_i = Q_i(x) \in k[x]$, $\deg Q_1 = 1$, $\deg Q_2 \leq 1$, $\deg Q_3 \leq 3$, $\deg Q_4 \leq 1$, $\deg Q_5 \leq 2$ and $\deg Q_6 = 4$. Eliminating z from the above two relations, we have

$$(y - Q_4)(y^2 - Q_2y - Q_3) = Q_1(Q_5y + Q_6),$$

that is to say,

$$y^3 + Q_7(x)y^2 + Q_8(x)y + Q_9(x) = 0$$

where $Q_i(x) \in k[x]$, $\deg Q_7(x) \leq 1$, $\deg Q_8(x) \leq 3$ and $\deg Q_9(x) = 5$. If C' is the curve corresponding to $k(x, y)$, then the genus g' of C' is less than or equal to $(5-1)(5-2)/2 = 6$ and C is a double covering of C' . The assumption $H(P) \ni 4, 8$ implies that $g' \geq 3$. Since $x, y, z \in K(C')$, we have $H(f(P)) \ni \langle 3, 5, 7 \rangle$ where $f: C \rightarrow C'$ is the morphism corresponding to

the inclusion $K(C') \subset K(C)$. Hence $g' = 4$ and C' is non-hyperelliptic. Q.E.D.

Lemma 2.6. (1) If $g \geq 22$ and if C is a double covering of a curve of genus 4, then we have one of the following : (a) $H(P) \ni 8, H(P) \ni 10, H(P) \ni 14$, (b) $H(P) \ni 8, 10, H(P) \ni 14, 16$.

(2) If $g \geq 23$, then the following are equivalent : (A) $H(P) \ni 8, H(P) \ni 10, H(P) \ni 14$ (resp. $H(P) \ni 8, 10, H(P) \ni 14, 16$), (B) C is a double covering of a non-hyperelliptic curve of genus 4 with a half canonical divisor D' (resp. with no half canonical divisor D') with $l(D') = 2$.

Proof. (1) Let $f: C \rightarrow C'$ be a double covering of a curve C' of genus 4. Applying Corollary 1.2 to this case we have $x \in K(C)$, hence $H(f(P))$ must be either $\langle 3, 5 \rangle$ or $\langle 3, 7, 8 \rangle$. If $H(f(P)) = \langle 3, 5 \rangle$ (resp. $\langle 3, 7, 8 \rangle$) then the case (a) (resp. (b)) occurs.

(2) First we show that (A) induces (B). Suppose that $H(P) \ni 8, H(P) \ni 10, H(P) \ni 14$. Then we obtain

$$H(P) = \{0, 6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 30, 31, \dots\}.$$

If y is a function with $(y)_\infty = 10P$, then we have a relation

$$y^3 = Q_1(x)y^2 + Q_2(x)y + Q_3(x)$$

where $Q_i(x) \in k[x]$, $\deg Q_1(x) \leq 1$, $\deg Q_2(x) \leq 3$ and $\deg Q_3(x) = 5$. Hence the curve C' corresponding to $k(x, y)$ is of genus $g' \leq 6$ and the natural map $f: C \rightarrow C'$ is a double covering. By Proposition 2.3, C is 6-gonal, which implies that $g' \geq 3$. If $g' = 3$, this is a contradiction to Lemma 2.5 (1). Since $H(P) \ni 6, 10$, we get $H(f(P)) \ni 3, 5$. Hence $g' \leq \#(N - \langle 3, 5 \rangle) = 4$. This shows that $g' = 4$ and $H(f(P)) = \langle 3, 5 \rangle$. Now if we set $D' = 3P'$, then we have $\deg 2D' = 6$ and $l(2D') = l(6P') = 4$, which means that D' is a half canonical divisor with $l(D') = 2$. Next, suppose that $H(P) \ni 8, 10$ and $H(P) \ni 14, 16$. Then we have

$$H(P) = \{0, 6, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 31, \dots\}.$$

Let y, z be functions with $(y)_\infty = 14P$ and $(z)_\infty = 16P$. In the same way as in the proof of Lemma 2.5 (2), we see that the curve C' corresponding to $k(x, y)$ is of genus $g' \leq 15$ and that the natural morphism $f: C \rightarrow C'$ is a double covering. It is easy to see that $g' = 4$ and $H(f(P)) = \langle 3, 7, 8 \rangle$. Assume that there exists a half canonical divisor D' with $l(D') = 2$. Then it is known that $W_3^2(C')$ consists of one point $j_3(D')$.³⁾ Hence we get $D' \sim 3f(P)$, which induces $3 = l(6f(P)) = l(2D') = 4$. This is a contradiction.

Lastly we show that (B) means (A). By (1) either (a) or (b) occurs. Let C' be a curve of genus 4 such that there is a double covering $f: C \rightarrow C'$. Then applying Corollary 1.2 (2) to the case $p = 2$, $g' = 4$, $g \geq 23$ and $K_0 = K(C')$, such a C' is uniquely determined up to isomorphisms. Hence (B) implies (A). Q.E.D.

If we rewrite Proposition 1.7 and Theorem 1.8 using the results in this section, then we get the Main Theorem which is stated in Introduction.

§ 3. On the Existence of Weierstrass Points whose First Non-gaps are Six.

Definition 3.1. Let n be an integer with $n \geq 2$. A numerical semigroup H is said to be n -cyclic if H starts with n and if there exists a triplet (C, T, P) where C is a curve, T is its automorphism of order n with $C/\langle T \rangle \cong \mathbf{P}^1$ and P is a fixed point of T with $H(P) = H$.

Remark 3.2. Let C be a curve with an automorphism T of order n satisfying $C/\langle T \rangle \cong \mathbf{P}^1$. Then it is known that C can be defined by an equation of the form

$$y^n = \prod_{j=1}^I (x - c_j)^{r_j}$$

where c_j 's are distinct elements of k and r_j 's are integers with $1 \leq r_j \leq n-1$ and $\sum_{j=1}^I r_j \equiv 0 \pmod n$.

In the final section we are concerned with $n=6$ and we use the following notation: C is defined by an equation of the form

$$y^6 = \prod_{i=1}^5 \prod_{j=1}^{n_i} (x - c_{ij})^i$$

where c_{ij} 's are distinct elements of k , and $6, \sum_{i=1}^5 in_i$ are relatively prime. Let $f: C \rightarrow \mathbf{P}^1$ denote the surjective morphism of degree 6 defined by sending any point P of C to $(1, x(P))$, and we set

$$\begin{aligned} f^{-1}((0, 1)) &= \{P_\infty\} \text{ and } f^{-1}((1, c_{ij})) = \{P_{ij}\} \\ &(\text{resp. } \{P_{ij}, P'_{ij}\}, \text{ resp. } \{P_{ij}, P'_{ij}, P''_{ij}\}) \text{ for } i=1, 5 \\ &(\text{resp. } i=2, 4, \text{ resp. } i=3) \text{ and } 1 \leq j \leq n_i. \end{aligned}$$

Then it follows from Hurwitz's theorem that the genus g of C is equal to

$$(5(n_1 + n_5 + 1) + 3n_3)/2 + 2(n_2 + n_4) - 5.$$

Moreover, we note that if H is 6-cyclic, then there are non-negative integers $n_i (1 \leq i \leq 5)$ such that $H = H(P_\infty)$. First we calculate the numerical semigroup $H(P_\infty)$. It is easy to see the following:

Lemma 3.3. We have

$$\begin{aligned} \text{div}(y) &= -(\sum_{i=1}^5 in_i)P_\infty + \sum_{i=1,5} i \sum_{j=1}^{n_i} P_{ij} \\ &+ \sum_{i=2,4} (i/2) \sum_{j=1}^{n_i} (P_{ij} + P'_{ij}) + \sum_{j=1}^{n_3} (P_{3j} + P'_{3j} + P''_{3j}). \end{aligned}$$

Proposition 3.4. Let

$$b_m = 6(-\sum_{i=1}^5 n_i[-mi/6]) - m \sum_{i=1}^5 in_i$$

for each $m \in \{1, \dots, 5\}$ where $[\]$ denotes the Gauss symbol. Then we have

$$b_m = \text{Min}\{h \in H \mid h \equiv m \pmod 6\}$$

for each $m \in \{1, \dots, 5\}$. Hence the semigroup $H(P_\infty)$ is generated by b_1, \dots, b_5 .

Proof. Since for $i=1, 5$ (resp. $i=2, 4$, resp. $i=3$) we have

$$\begin{aligned} \text{div}(x - c_{ij}) &= -6P_\infty + 6P_{ij} \text{ (resp. } -6P_\infty + 3P_{ij} + 3P'_{ij}, \\ &\text{resp. } -6P_\infty + 2P_{ij} + 2P'_{ij} + 2P''_{ij}), \end{aligned}$$

we see easily that for each $m \in \{1, \dots, 5\}$

$$\left(\prod_{i=1}^5 \prod_{j=1}^{n_i} (x - c_{ij})^{-\lfloor -mi/6 \rfloor} / y^m \right)_\infty = b_m P_\infty.$$

Hence $H(P_\infty)$ contains the semigroup $H = \langle 6, b_1, \dots, b_5 \rangle$. Now we claim that $g(H) \leq g = g(H_\infty)$. Let $d = \sum_{i=1}^5 in_i$. Since the 6-residues of md 's ($1 \leq m \leq 5$) are distinct, we obtain

$$g(H) \leq \sum_{m=1}^5 ((-\sum_{i=1}^5 n_i \lfloor -mi/6 \rfloor) + \lfloor -md/6 \rfloor).$$

Computations show that

$$-\sum_{m=1}^5 \lfloor -mi/6 \rfloor = 5 \text{ (resp. 7, resp. 3i) for } i=1 \text{ (resp. } i=2, \text{ resp. } 3 \leq i \leq 5).$$

Moreover, we have

$$\sum_{m=1}^5 \lfloor -md/6 \rfloor = -5(d+1)/2$$

because 6 and d are relatively prime. Hence we get $g(H) \leq g$. From $g(H) \leq g(H(P_\infty))$ and $H \subseteq H(P_\infty)$ it follows that for each $m \in \{1, \dots, 5\}$ b_m is equal to the minimum of elements h of H with $h \equiv m \pmod{6}$ and that $H(P_\infty) = H$. Q.E.D.

Computations of the Gauss symbols $\lfloor -mi/6 \rfloor$ ($1 \leq i \leq 5$, $1 \leq m \leq 5$) show the following :

Corollary 3.5. *We have*

$$H(P_\infty) = \langle 6, 5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5, 4n_1 + 2n_2 + 4n_4 + 2n_5, \\ 3n_1 + 3n_3 + 3n_5, 2n_1 + 4n_2 + 2n_4 + 4n_5, n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 \rangle.$$

Using Proposition 3.4 and Corollary 3.5 we can give examples of the Main Theorem.

Example 3.6. In this example, we set $H = H(P_\infty)$.

- (1) If $n_1 + n_4 = 1$ (resp. 2), $n_2 + n_5 = 2$ (resp. 1) and $n_3 \geq 2$, then H contains 8, 10 and starts with 6.
- (2) If $n_1 + n_3 + n_5 = 3$ and $n_2 + n_4 \geq 3$, then H contains 9 and starts with 6.
- (3) If $n_1 + n_4 = 0$ (resp. 4), $n_2 + n_5 = 4$ (resp. 0) and $n_3 \geq 2$, then H starts with 6 and $H \ni 8$, $H \ni 10$.
- (4) If $n_1 + n_4 = 1$ (resp. 3), $n_2 + n_5 = 3$ (resp. 1) and $n_3 \geq 2$, then H starts with 6 and $H \ni 8$, $H \ni 10$, 14.
- (5) If $n_1 + n_4 = 2$ (resp. 3), $n_2 + n_5 = 3$ (resp. 2) and $n_3 \geq 2$, then H starts with 6 and $H \ni 8$, 10, $H \ni 14$, 16.

In all cases, the set of integers

$$g = (5(n_1 + n_5 + 1) + 3n_3)/2 + 2(n_2 + n_4) - 5$$

satisfying the condition, contains the set of integers which are larger than d for some positive integer d .

Lastly we give numerical semigroups starting with 6 which are not 6-cyclic and which curves admit.

Example 3.7. (1) If $H = \langle 6, 6p+1, 6q+5 \rangle$ where p and q are positive integers with $p \leq q \leq 5p-1$, then H is not 6-cyclic and there exists a curve admitting H .

(2) If $H = \langle 6, 6p+1, 6q+5, 6r+2 \rangle$ where p , q and r are positive integers with $p \leq q < r \leq 2p-1$, then H is not 6-cyclic and there exists a curve admitting H .

Proof. For each integer m with $1 \leq m \leq 5$, we set

$$a_m = \text{Min}\{h \in H \mid h \equiv m \pmod{6}\}.$$

If H is 6-cyclic, i.e., $H = H(P_\infty)$, then by Proposition 3.4 and Corollary 3.5 we easily see that

$$n_2 + n_4 = (a_1 - 2a_3 + a_5)/6.$$

In the case (1) we have

$$a_1 = 6p + 1, \quad a_3 = 18p + 3 \quad \text{and} \quad a_5 = 6q + 5.$$

If $H = H(P_\infty)$, then by the above we have

$$0 \leq n_2 + n_5 = (6p + 1 - 36p - 6 + 6q + 5)/6 = -5p + q \leq -1.$$

This is a contradiction. Hence H is not 6-cyclic. On the other hand, it is known that for any numerical semigroup H generated by three elements, there exists a curve admitting H .⁶⁾

In the case (2) we have

$$a_1 = 6p + 1, \quad a_3 = 6(p + r) + 3 \quad \text{and} \quad a_5 = 6q + 5.$$

It H is 6-cyclic, then

$$0 \leq n_2 + n_4 = (6p + 1 - 12(p + r) - 6 + 6q + 5)/6 = -p - 2r + q < 0.$$

This is a contradiction. Now we set

$$d_1 = 6, \quad d_2 = 6p + 1, \quad d_3 = 6q + 5 \quad \text{and} \quad d_4 = 6r + 2.$$

Then we have relations

$$\begin{aligned} (p + q + 1)d_1 &= d_2 + d_3, \quad 2d_4 = (2p - r)d_1 + d_4, \\ 3d_3 &= (3q - p - r + 2)d_1 + d_2 + d_4 \quad \text{and} \quad 2d_4 = (2r - 2q - 1)d_1 + 2d_3. \end{aligned}$$

Moreover, we have

$$\begin{vmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \end{vmatrix} = 6 = d_1.$$

Hence H is 1-neat, which implies that there is a curve admitting H .⁶⁾

Q.E.D.

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