Numerical Semigroups and Non-gaps of Weierstrass Points

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Abstract

Let $\boldsymbol{N}$ be the additive semigroup of non-negative integers. A subsemigroup H of $\boldsymbol{N}$ is called a *numerical semigroup* if the complement of H in $\boldsymbol{N}$ is finite. Then the following is a long-standing problem. Find a necessary and sufficient condition on a numerical semigroup H such that H is equal to the set of non-gaps of some point of a curve (in this case H is said to be *Weierstrass*), where a curve means a complete non-singular 1-dimensional algebraic variety over an algebraically closed field of characteristic 0. In this paper we give the list of known Weierstrass numerical semigroups, and we concretely describe p-cyclic numerical semigroups which are typical Weierstrass numerical semigroups. Moreover, infinite examples of non-Weierstrass numerical semigroups are given.

Notation

Throughout this paper we will use the following notation without further warning. $\boldsymbol{N}$ denotes the additive semigroup of non-negative integers. Let H be a numerical semigroup, i.e. a subsemigroup of $\boldsymbol{N}$ such that the complement $\boldsymbol{N}-H$ of H in $\boldsymbol{N}$ is finite. We denote by $g(H)$ (resp. $M(H)$) the genus of H (resp. the minimal set of generators for H). $C(H)$ denotes the conductor of H, i.e. the minimum of integers c such that $c+\boldsymbol{N}\subseteqq H$. If $\boldsymbol{N}-H=\{n_1, \cdots, n_g\}$, then we set

$$w(H)=\sum_{i=1}^{g}(n_i-i),$$

which is called the *weight* of H. Let $M(H)=\{a_1, \cdots, a_n\}$. Then I_H denotes the kernel of the k-algebra homomorphism $\varphi: k[X]=k[X_1, \cdots, X_n]\rightarrow k[t]$ defined by $\varphi(X_i)=t^{a_i}$ where $k[X]$ and $k[t]$ are polynomial rings over k, and $\mu(H)$ denotes the least number of generators for the ideal I_H. When we set $C_H=\operatorname{Spec} k[X]/I_H$, we denote by $T^1_{C_H}=\bigoplus_{l\in\boldsymbol{Z}} T^1_{C_H}(l)$ the k-vector space of first order deformations of C_H with a natural graded structure. Moreover, $a_1(H)$ denotes the first non-gap of H, i.e. the least positive integer h of H. For elements $a_1, \cdots, a_n$ of $\boldsymbol{N}$, $\langle a_1, \cdots, a_n\rangle$ denotes the subsemigroup of $\boldsymbol{N}$ generated by $a_1, \cdots, a_n$.

§1. On Weierstrass numerical semigroups.

Let H be a numerical semigroup. Then using the deformation theory on algebraic varieties with $\boldsymbol{G}_m$-action, Pinkham constructed a moduli space M_H which classifies the set of isomorphic classes of pairs (C, P) consisting of a curve C together with its point P such that

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$H(P)=H$[1]. Using the Pinkham's construction of M_H, some mathematicians showed that for some H, M_H is non-empty, i.e. H is Weierstrass. In fact, it is known that H is Weierstrass in the following cases:

(1) H is a complete intersection, i.e. $\#M(H)=n$ and $\mu(H)=n-1$,

(2) H is a special almost complete intersection[2],

(3) H is negatively graded, i.e. $T^1_{C_H}(l)=0$ for $l>0$[1],[3],

(4) $w(H)<a_1(H)$[4],

(5) $g(H)\leqq 6$,

(6) $a_1(H)\leqq 4$[5],

(7) $\#M(H)\leqq 3$.

Moreover, to state results in the case $\#M(H)=4$ we prepare some definition. Let $M(H)=\{a_1, a_2, a_3, a_4\}$. Then H is said to be 1-*neat* if a_1, a_2, a_3 and a_4 satisfy

$$\alpha_i a_i=\sum_{j\neq i}\alpha_{ij}a_j \quad \text{with } 0\leqq \alpha_{ij}<\alpha_j, \quad \text{for} \quad 1\leqq i\leqq 4 \quad \text{and}$$

$$\sum_{i\neq j}\alpha_{ij}=\alpha_j \quad \text{for} \quad 1\leqq j\leqq 4$$

where $\alpha_i=\text{Min}\,\{\alpha\in\boldsymbol{N}\mid\alpha>0, \quad \alpha a_i\in\sum_{j\neq i}\boldsymbol{N}a_j\}$, and if

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_2 & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_3 \end{vmatrix} = a_4$$

(8) $\#M(H)=4$ and H is symmetric, i.e. $C(H)=2g(H)$[6],[7].

(9) $\#M(H)=4$ and H is $1-$neat[8].

(10) H is n-cyclic, i.e. $a_1(H)=n$ and there exists a triplet (C, T, P) where C is a curve, T is its automorphisrm of order n with $C/\langle T\rangle\simeq\boldsymbol{P}^1$ and P is a fixed point of T with $H(P)=H$.

Since the above condition (10) is abstract, we describe p-cyclic numerical semigroups in the case of prime p.

Proposition 1. *Let H be a p-cyclic numerical semigroup. Then there exist non-negative integers $i_1, \cdots, i_{p-1}$ with $p\nmid\sum_{q=1}^{p-1}qi_q$, such that*

$$H=\boldsymbol{N}p+\boldsymbol{N}b+\sum_{m=1}^{p-2}\boldsymbol{N}(-p\sum_{q=1}^{p-1}[-mq/p]i_q-mb)$$

where $b=\sum_{q=1}^{p-1}qi_q$ and [ ] denotes the Gauss symbol.

Proof. Let $i_0=0$ and let $i_1, \cdots, i_{p-1}$ be non-negative integers with $p\nmid\sum_{q=1}^{p-1}qi_q$. For any $0\leqq q\leqq p-1$, we set $I_q=\sum_{l=0}^{q}i_l$. Let C be the curve defined by an equation of the form

$$y^p=\prod_{q=1}^{p-1}\prod_{j=I_{q-1}+1}^{I_q}(x-c_j)^q$$

where $c_1, \cdots, c_{I_{p-1}}$ are distinct elements of k. Let $f: C\rightarrow\boldsymbol{P}^1$ denote the surjective morphism of degree p defined by sending any point P of C to $(1, x(P))$, and we set $f^{-1}((0,1))=\{P_\infty\}$.

Then there are non-negative integers $i_1, \cdots, i_{p-1}$ with $p \nmid \sum_{q=1}^{p-1} qi_q$, such that $H=H(P_\infty)$, because H is p-cyclic. Hence it suffices to show that $H(P_\infty)=H_0$ where

$$H_0=\boldsymbol{N}p+\boldsymbol{N}b+\sum_{m=1}^{p-2}\boldsymbol{N}(-p\sum_{q=1}^{p-1}[-mq/p]i_q-mb)$$

and $b=\sum_{q=1}^{p-1} qi_q$. First we show that $H(P_\infty)\supseteqq H_0$. Now

$$\begin{aligned}\operatorname{div}(y^p)&=\operatorname{div}(\prod_{q=1}^{p-1}\prod_{j=I_{q-1}+1}^{I_q}(x-c_j)^q)\\&=p\sum_{q=1}^{p-1}q\sum_{j=I_{q-1}+1}^{I_q}P_j-pbP_\infty\end{aligned}$$

where we set $f^{-1}((1, c_j))=\{P_j\}$. Hence we get

$$\operatorname{div}(y)=\sum_{q=1}^{p-1}q\sum_{j=I_{q-1}+1}^{I_q}P_j-bP_\infty.$$

For any $1\leqq m\leqq p-2$, we put

$$z_m=\prod_{q=1}^{p-1}\prod_{j=I_{q-1}+1}^{I_q}(x-c_j)^{-[-mq/p]}.$$

Then we get

$$\operatorname{div}(z_m/y^m)=\sum_{q=1}^{p-1}(-[-mq/p]p-mq)\sum_{j=I_{q-1}+1}^{I_q}P_j-(-p(\sum_{q=1}^{p-1}i_q[-mq/p])-mb)P_\infty.$$

Since $-[-mq/p]p-mq>0$, we have

$$H(P_\infty)\ni -p(\sum_{q=1}^{p-1}i_q[-mq/p])-mb.$$

Moreover, $(x)_\infty=pP_\infty$ and $(y)_\infty=bP_\infty$. Hence we get $H(P_\infty)\supseteqq H_0$. Since residues of b and $-mb$'s $(1\leqq m\leqq p-2)$ modulo p are distinct, we get

$$g(H_0)\leqq[b/p]+\sum_{m=1}^{p-2}([-mb/p]-\sum_{q=1}^{p-1}[-mq/p]i_q).$$

We set

$$G=[b/p]+\sum_{m=1}^{p-2}([-mb/p]-\sum_{q=1}^{p-1}[-mq/p]i_q),$$

$$G_1=[b/p]+\sum_{m=1}^{p-2}[-mb/p]$$

and

$$G_2=-\sum_{m=1}^{p-2}\sum_{q=1}^{p-1}[-mq/p]i_q.$$

Then we get

$$G_1=-(p-2)b+\sum_{m=1}^{p-1}[mb/p]$$

and

$$G_2=(p-2)I_{p-1}+\sum_{m=1}^{p-2}\sum_{q=1}^{p-1}[mq/p]i_q.$$

Moreover, we have $\sum_{m=1}^{p-1}[mb/p]=(p-1)(b-1)/2$ and

$$\begin{aligned}\sum_{q=1}^{p-1}i_q\sum_{m=1}^{p-2}[mq/p]&=\sum_{q=1}^{p-1}i_q((p-1)(q-1)/2-q+1)\\&=(p-1)(b-I)/2-b+I,\end{aligned}$$

where we set $I=I_{p-1}$. Hence

$$G=G_1+G_2$$
$$=(p-2)(I-b)+(p-1)(b-1)/2+(p-1)(b-I)/2-b+I$$
$$=(p-2)(I-1)/2.$$

On the other hand by Hurwitz's Theorem we obtain

$$2g(H(P_\infty))-2=-2p+(p-1)(I+1),$$

which implies that

$$g(H(P_\infty))=(p-1)(I-1)/2=G\geqq g(H_0).$$

Since $H(P_\infty)\supseteqq H_0$ and $g(H(P_\infty))\geqq g(H_0)$, we conclude that $H(P_\infty)=H_0$. *Q.E.D.*

In the 4-cyclic (resp. 6-cyclic) case, see 5) (resp. 9)) in References.

§2. Examples of non-Weierstrass numerical semigroups

It is known that the numerical semigroup ⟨13, 14, 15, 16, 17, 18, 20, 22, 23⟩ is non-Weierstrass. In this section we will give infinite examples of non-Weierstrass numerical semigroups containing the above.

Lemma 2. *Let H be a Weierstrass numerical semigroup of genus* $g\geqq 2$, *and let* $\{1=n_1<n_2<\cdots<n_g\}$ *be the set of gaps of H. Then the number of the set of* $(n_i-1)+(n_j-1)$, $1\leqq i,\ j\leqq g$ *is less than or equal to* $3g-3$.

Proof. Since H is Weierstrass, there are a curve C and its point P such that $H(P)=H$. Let $dfk(C)$ be the set of differentials on C of the first kind and v_P the normalized valuation associated to P. Since $1=n_1<n_2<\cdots<n_g$ is the gap sequence at P, for each i with $1\leqq i\leqq g$ there is a non-zero element d_i of $dfk(C)$ such that $v_P(d_i)=n_i-1$. For each i with $1\leqq i\leqq g$ let f_i be a meromorphic function on C such that $d_i=f_id_1$. Then we get

$$v_P(f_i)=v_P(d_i)-v_P(d_1)=(n_i-1)-(n_1-1)=n_i-1.$$

Since f_i's are contained in $L(\mathrm{div}(d_1))$ where for any divisor D, $L(D)$ denotes the k-vector space consisting of meromorphic functions f on C such that $\mathrm{div}(f)\geqq -D$, we obtain $f_if_j\in L(2\,\mathrm{div}(d_1))$ for any i and any j. In view of $v_P(f_if_j)=(n_i-1)+(n_j-1)$, we must have

$$\#\{\cdots,\ (n_i-1)+(n_j-1),\ \cdots\}\leqq\dim_k L(2\,\mathrm{div}(d_1))=3g-3. \qquad Q.E.D.$$

Proposition 3. *Let N and M be positive integers such that* $N\geqq 2M+3$ *and* $M\geqq 2$. *We set*

$H=\langle N,\ N+1,\ \cdots,\ N+N-2(M+1),\ 2N-2M,\ 2N-2(M-1),\ \cdots,\ 2N-2\cdot 3,\ 2N-2\cdot 2,\ 2N-3\rangle$.

Then

(0) $a_1(H)=N$ *and* $\#M(H)=N-M-1$.

(1) $g(H)=N+M$.

(2) *If* $N\geqq 4M+1$ *and* $M\geqq 3$, *then H is non-Weierstrass.*

Proof. (0) is obvious. Note that for each $2\leqq i\leqq M$ we have $3N-(2i+1)=(N+1)+(N+N-2(i+1))$. Moreover $3N-2=(N+1)+(2N-3)$. If $N\geqq 2M+4$, then $3N-1=(N+2)+(2N-3)$. If $N=2M+3$, then $3N-1=(2N-2M)+2N-4$. Hence the gaps of H are 1, 2, $\cdots$, $N-2(M+1)$, $N-(2M+1)$, $2N-(2M+1)$, $N-2M$, $N-(2(M-1)+1)$, $2N$

$-(2(M-1)+1)$, $N-2(M-1)$, $\cdots$, $N-6$, $N-5$, $2N-5$, $N-4$, $N-3$, $N-2$, $2N-2$, $N-1$, $2N-1$. The above implies (1), i.e. $g(H)=N+M$. Now if we denote by $\{n_1, n_2, \cdots, n_{g(H)-1}\}$ the set of gaps of H, then we have

$$\{n_1-1, n_2-1, \cdots, n_{g(H)}-1\}$$
$$=\{0, 1, \cdots, N-2, 2N-2(M+1), 2N-2M, \cdots, 2N-2\cdot 2, 2N-3, 2N-2\}.$$

We will calculate the number of the set

$$G=\{(n_i-1)+(n_j-1) \mid 1\leqq i, j\leqq g(H)\}.$$

The sums $0+i$, $N-2+i$, $2N-2+i$ $(0\leqq i\leqq N-2)$ and $0+2N-3$ imply that G contains 0, 1, 2, $\cdots$, $3N-5$, $3N-4$. The rest of elements of G are of type $i+j$ where $i=2N-2l$ $(M+1\geqq l\geqq 3)$ or $2N-3$ or $2N-2$, and $j=2N-2l$ $(M+1\geqq l\geqq 3)$ or $2N-3$ or $2N-2$. Hence it suffices to calculate the number of the set $S=\{q+r \mid q, r\in T\}$ where $T=\{2, 3, 6, 8, 10, \cdots, 2M, 2(M+1)\}$. It is easy to see that $\#S=3M+1$. Hence we get $\#G=3N-3+3M+1=3N+3M-2>3N+3M-3=3g(H)-3$. By Lemma 2, we conclude that H is non-Weierstrass.

Q.E.D.

Corollary 4. (1) *For any* $g\geqq 16$, *there is a non-Weierstrass numerical semigroup of genus* g.

(2) *For any* $a\geqq 13$, *there is a non-Weierstrass numerical semigroup* H *with* $a_1(H)=a$.

(3) *For any* $l\geqq 9$, *there is a non-Weierstrass numerical semigroup* H *with* $\#M(H)=l$.

Proof. In Proposition 3, let $M=3$. Then $g(H)=N+3\geqq 16$, $a_1(H)=N\geqq 13$ and $\#M(H)=N-4\geqq 9$.

Q.E.D.

In view of (5), (6), (7) in §1 and Corollary 4, we set open problems as follows:

Problem. (1) For each $7\leqq g\leqq 15$, is there a non-Weierstrass numerical semigroup of genus g?

(2) For each $5\leqq a\leqq 12$, is there a non-weirstrass numerical semigroup H with $a_1(H)=a$?

(3) For each $4\leqq l\leqq 8$, is there a non-Weierstrass numerical semigroup H with $\#M(H)=l$?

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