# On Weierstrass Points Whose First Non-gaps are Five

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**Abstract**

Let $N$ be the additive semigroup of non-negative integers. A subsemigroup $H$ of $N$ is called a *numerical semigroup* if the complement of $H$ in $N$ is finite. If $a$ is the least positive integer which belongs to $H$, then we say that $H$ *starts with* $a$. In this paper we give sufficient conditions on a numerical semigroup $H$ starting with 5 such that $H$ is equal to the set of nongaps of some point of a curve (in this case $H$ is said to be *Weierstrass*), where a *curve* means a complete non-singular 1-dimensional algebraic variety over an algebraically closed field of characteristic 0.

## Introduction

Let $H$ be a numerical semigroup. Then the number of the set $N$-$H$ is called the *genus* of $H$, which is denoted by $g(H)$. Let $C$ be a curve. For any point $P$ of $C$, $H(P)$ denotes the set of non-gaps $n$ at $P$, i.e. non-negative integers $n$ satisfying

$$H^o(C,\ O_c((n-1)P))\subset H^o(C,\ O_c(nP)).$$

Then $H(P)$ becomes a numerical semigroup. If $H(P)$ starts with $a$, then by abuse of language $a$ is called the *first non-gap* of $P$.

Now there is a long-standing problem as follows: describe a necessary and sufficient condition for a numerical semigroup $H$ to be Weierstrass. If $H$ starts with 2, then it is equal to $H(P)$ where $P$ is a Weierstrass point of a hyperelliptic curve. If $H$ starts with 3, then it is Weierstrass[1]. If $H$ starts with 4, then it is also Weierstrass[2]. In this paper we will describe sufficient conditions on a numerical semigroup $H$ starting with 5 to be Weierstrass. In fact, our result is the following: *let $H$ be a numerical semigroup starting with 5 and let $M(H)$ be the minimal set of generators for $H$.*

(0) *If $\#M(H)\leqq 3$, then $H$ is Weierstrass*[3].

(1) *Let $M(H)=\{5<a_1<a_2<a_3\}$. Suppose that one of the following holds:*

a) $2a_3\equiv a_1+a_2 \bmod 5$,

b) $2a_2\equiv a_1+a_3 \bmod 5$ *and* $2a_2\geqq a_1+a_3$,

c) $2a_1\equiv a_2+a_3 \bmod 5$, $a_1+a_2\equiv 0 \bmod 5$ *and* $2a_3\geqq 2a_1+a_2$,

d) $2a_1\equiv a_2+a_3 \bmod 5$, $a_1+a_3\equiv 0 \bmod 5$ *and* $2a_2\geqq 2a_1+a_3$.

*Then $H$ is Weierstrass.*

(2) *Let $M(H)=\{5,\ 5q_1+1,\ 5q_2+2,\ 5q_3+3,\ 5q_4+4\}$. Suppose that one of the following holds:*

a) $q_1+q_4=q_2+q_3$,

b) $2q_1-q_3-q_4-1\geqq 0$ *and* $2q_4+1-q_1-q_2\geqq 0$,

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c) $2q_2 - q_1 - q_3 \geqq 0$ *and* $2q_3 - q_2 - q_4 \geqq 0$.

*Then* $H$ *is Weierstrass.*

Using the above result we will show that any numerical semigroup of genus $g \leqq 7$ at least except $\langle 5, 7, 11, 13\rangle$ is Weierstrass, where for positive integers $a_0, a_1, \cdots, a_{n-1}$, $\langle a_0, a_1, \cdots, a_{n-1}\rangle$ denotes the semigroup generated by $a_0, a_1, \cdots, a_{n-1}$. Moreover, we will investigate whether a numerical semigroup of genus $8 \leqq g \leqq 10$ starting with 5 is Weierstrass.

## §1. 5-cyclic numerical semigroups.

**Definition 1.1.** Let $n$ be an integer with $n \geqq 2$. A numerical semigroup $H$ is said to be *$n$-cyclic* if $H$ starts with $n$ and if there exists a triplet $(C, T, P)$ where $C$ is a curve, $T$ is its automorphisn of order $n$ with $C/\langle T\rangle \cong \boldsymbol{P}^1$ and $P$ is a fixed point of $T$ with $H(P) = H$, where $\langle T\rangle$ is the group generated by $T$.

In this section first we will give a description of a 5-cyclic numerical semigroup $H$. Using this we will show that any $H$ with $\# M(H) = 4$ is not 5-cyclic and give a necessary and sufficient condition for $H$ to be 5-cyclic.

**Lemma 1.2.** *Let $H$ be a numerical semigroup starting with 5. Then the following are equivalent*:

(1) *$H$ is 5-cyclic,*

(2) *there are non-negative integers $n_1$, $n_2$, $n_3$ and $n_4$ with $5 \nmid n_1 + 2n_2 + 3n_3 + 4n_4$ such that*

$$H = \langle 5,\ n_1 + 2n_2 + 3n_3 + 4n_4,\ 2n_1 + 4n_2 + n_3 + 3n_4,\ 3n_1 + n_2 + 4n_3 + 2n_4,\\ 4n_1 + 3n_2 + 2n_3 + n_4\rangle.$$

*Proof.* $H$ is 5-cyclic if and only if the following situation holds : let C be a curve defined by an equation of the form

$$y^5 = \prod_{i=1}^{4} \prod_{j=1}^{n_i} (x - c_{ij})^i$$

where $c_{ij}$'s are distinct elements of $k$, and $n_1$, $n_2$, $n_3$ and $n_4$ are non-negative integers with $5 \nmid \sum_{i=1}^{4} in_i$. Let $f: C \to \boldsymbol{P}^1$ be the surjective morphism of degree 5 defined by sending any point $P$ of $C$ to $(1, x(P))$ and let $f^{-1}((0, 1)) = \{P_\infty\}$. Then $H = H(P_\infty)$. Therefore it suffices to calculate $H(P_\infty)$. Hence the proof is complete[4]. *Q.E.D.*

**Remark 1.3.** Let $H$ be a numerical semigroup starting with $a$. Then we define a set of generators for $H$, which is denoted by $S(H)$, inductively as follows. Let $s_0 = a$. If $s_0 < s_1 < \cdots < s_i$ have been chosen and $i < a-1$, then $s_{i+1}$ is defined by the least integer in $H$ having $a$-residue distinct from those of $s_0, s_1, \cdots, s_i$. Then we set $S(H) = \{s_0, s_1, \cdots, s_{a-1}\}$. In the case of Lemma 1.2 we have $S(H) = \{5, b_1, b_2, b_3, b_4\}$ where we set $b_1 = n_1 + 2n_2 + 3n_3 + 4n_4$, $b_2$

$=2n_1+4n_2+n_3+3n_4$, $b_3=3n_1+n_2+4n_3+2n_4$ and $b_4=4n_1+3n_2+2n_3+n_4$.

**Propositon 1.4.** *If $H$ is a numerical semigroup starting with 5 and satisfying $\#M(H)=4$, then $H$ is not 5-cyclic.*

*Proof.* Suppose that $H$ is 5-cyclic. We use the notation in Remark 1.3. In view of $\#M(H)=4$ there is a unique $i\in\{1, 2, 3, 4\}$ such that $b_i\notin M(H)$. In the case $i=1$ we have $M(H)=\{5, b_2, b_3, b_4\}$. Hence $b_1=b_2+b_4$ or $b_1=2b_3$. If $b_1=b_2+b_4$, then

$$n_1+2n_2+3n_3+4n_4=6n_1+7n_2+3n_3+4n_4,$$

which implies that $n_1=n_2=0$. Hence

$$\begin{aligned} H&=\langle 5,\ 3n_3+4n_4,\ n_3+3n_4,\ 4n_3+2n_4,\ 2n_3+n_4\rangle \\ &=\langle 5,\ n_3+3n_4,\ 2n_3+n_4\rangle, \end{aligned}$$

which contradicts $\#M(H)=4$. If $b_1=2b_3$, then

$$n_1+2n_2+3n_3+4n_4=6n_1+2n_2+8n_3+4n_4,$$

which implies that $n_1=n_3=0$. Hence

$$\begin{aligned} H&=\langle 5,\ 2n_2+4n_4,\ 4n_2+3n_4,\ n_2+2n_4,\ 3n_2+n_4\rangle \\ &=\langle 5,\ n_2+2n_4,\ 3n_2+n_4\rangle, \end{aligned}$$

which contradicts $\#M(H)=4$. Similarly, in the cases $i=2, 3, 4$ we have a contradiction.

*Q.E.D.*

**Proposition 1.5.** *Let $H$ be a numerical semigroup starting with 5 and let $S(H)=\{5, 5q_1+1, 5q_2+2, 5q_3+3, 5q_4+4\}$. Then $H$ is 5-cyclic if and only if $q_1+q_4=q_2+q_3$.*

*Proof.* If $H$ is 5-cyclic, then $b_1+b_4=b_2+b_3$ where $b_i$'s are as in Remark 1.3. This implies that $q_1+q_4=q_2+q_3$. Conversely we suppose that $q_1+q_4=q_2+q_3$. Consider the following simultaneous linear equations with unknowns $n_1$, $n_2$, $n_3$, $n_4$:

$$\begin{cases} n_1+2n_2+3n_3+4n_4=5q_1+1 \\ 2n_1+4n_2+n_3+3n_4=5q_2+2 \\ 3n_1+n_2+4n_3+2n_4=5q_3+3 \\ 4n_1+3n_2+2n_3+n_4=5q_4+4. \end{cases}$$

In view of $q_1+q_4=q_2+q_3$ these are equivalent to the following:

$$\begin{cases} n_1=n_4-3q_1+q_2+2q_3+1 \\ n_2=-n_4+q_1+q_2-q_3 \\ n_3=-n_4+2q_1-q_2. \end{cases}$$

Hence the solutions of the above are

$$(n_1,\ n_2,\ n_3,\ n_4)=(-3q_1+q_2+2q_3+1,\ q_1+q_2-q_3,\ 2q_1-q_2,\ 0)+\boldsymbol{R}(1,\ -1,\ -1,\ 1).$$

By Lemma 1.2 it suffices to find a solution consisting of non-negative integers. In the case $2q_2\geqq q_1+q_3$ we may take

$$(n_1, n_2, n_3, n_4)=(-q_1+2q_3+1, -q_1+2q_2-q_3, 0, 2q_1-q_2)$$

as a non-negative integer solution. In the case $2q_2<q_1+q_3$ we may take

$$(n_1, n_2, n_3, n_4)=(-2q_1+2q_2+q_3+1, 0, q_1-2q_2+q_3, q_1+q_2-q_3)$$

as a non-negative integer solution. *Q.E.D.*

## §2. Numerical semigroups $H$ starting with 5 and satisfying $\#M(H)=4$.

In this section we will give sufficient conditions for a numerical semigroup $H$ starting with 5 and satisfying $\#M(H)=4$ to be Weierstrass. Let $M(H)=\{a_0=5, a_1, a_2, a_3\}$. First we investigate minimal relations among $a_0$, $a_1$, $a_2$ and $a_3$.

**Lemma 2.1.** *Let* $M(H)=\{a_0=5<a_1<a_2<a_3\}$. *We set*

$$\alpha_i=\mathrm{Min}\{\alpha\in\boldsymbol{N}\geqq 2 \mid \alpha a_i\in\langle a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_3\rangle\}$$

*for all* $0\leqq i\leqq 3$. *Then we have the following.*

*Case A.1 (resp. Case A.2)*: $2a_3\equiv a_1+a_2 \bmod 5$ *and* $a_2+a_3\equiv 0 \bmod 5$ (*resp.* $a_1+a_3\equiv 0 \bmod 5$). *Then*

$$\alpha_0=\frac{a_2+a_3}{5}\left(resp.\ \frac{a_1+a_3}{5}\right)\ and\ \alpha_1=\alpha_2=\alpha_3=2.$$

*Case B.1 (resp. Case B.2)*: $2a_2\equiv a_1+a_3 \bmod 5$ *and* $a_2+a_3\equiv 0 \bmod 5$ (*resp.* $a_1+a_2\equiv 0 \bmod 5$). *If* $2a_2\geqq a_1+a_3$, *then*

$$\alpha_0=\frac{a_2+a_3}{5}\left(resp.\ \frac{a_1+a_2}{5}\right)\ and\ \alpha_1=\alpha_2=\alpha_3=2.$$

*Case C*: $2a_1\equiv a_2+a_3 \bmod 5$ *and* $a_1+a_2\equiv 0 \bmod 5$. *If* $2a_3\geqq 2a_1+a_2$, *then*

$$\alpha_0=\frac{a_1+a_2}{5},\ \alpha_1=3\ and\ \alpha_2=\alpha_3=2.$$

*Case D*: $2a_1\equiv a_2+a_3 \bmod 5$ *and* $a_1+a_3\equiv 0 \bmod 5$. *If* $2a_2\geqq 2a_1+a_3$, *then*

$$\alpha_0=\frac{a_1+a_3}{5},\ \alpha_1=3\ and\ \alpha_2=\alpha_3=2.$$

*Proof.* In the Case A.1, $2a_1\equiv a_3 \bmod 5$ and $2a_2\equiv a_1 \bmod 5$. Hence we get

$$\frac{a_2+a_3}{5}a_0=a_2+a_3,\ 2a_1=\frac{2a_1-a_3}{5}a_0+a_3,$$

$$2a_2=\frac{2a_2-a_1}{5}a_0+a_1,\ 2a_3=\frac{2a_3-a_1-a_2}{5}a_0+a_1+a_2.$$

Permuting suffixes 1 and 2 in the Case A.1 we get the result in the Case A.2. In the Case B.1, $2a_1\equiv a_2 \bmod 5$ and $2a_3\equiv a_1 \bmod 5$. Hence we get

$$\frac{a_2+a_3}{5}a_0=a_2+a_3,\ 2a_1=\frac{2a_1-a_2}{5}a_0+a_2,$$
$$2a_2=\frac{2a_2-a_1-a_3}{5}a_0+a_1+a_3,\ 2a_3=\frac{2a_3-a_1}{5}+a_1.$$

Permuting suffixes 1 and 3 in the Case B.1 we get the result in the Case B.2. In the Case C, we obtain $3a_1\equiv a_3 \bmod 5$, $2a_2\equiv a_3 \bmod 5$ and $2a_3\equiv 2a_1+a_2 \bmod 5$. Hence we get

$$\frac{a_1+a_2}{5}a_0=a_1+a_2,\ 3a_1=\frac{3a_1-a_3}{5}a_0+a_3,$$
$$2a_2=\frac{2a_2-a_3}{5}a_0+a_3,\ 2a_3=\frac{2a_3-2a_1-a_2}{5}a_0+2a_1+a_2.$$

In the Case D, we obtain $3a_1\equiv a_2 \bmod 5$, $2a_3\equiv a_2 \bmod 5$ and $2a_2\equiv 2a_1+a_3 \bmod 5$. Hence we get

$$\frac{a_1+a_3}{5}a_0=a_1+a_3,\ 3a_1=\frac{3a_1-a_2}{5}a_0+a_2,$$
$$2a_2=\frac{2a_2-2a_1-a_3}{5}a_0+2a_1+a_3 \text{ and } 2a_3=\frac{2a_3-a_2}{5}a_0+a_2.$$

*Q.E.D.*

**Theorem 2.2.** *Let $H$ be a numerical semigroup with $M(H)=\{a_0=5<a_1<a_2<a_3\}$. Suppose that one of the following holds.*

(A) $2a_3\equiv a_1+a_2 \bmod 5$,

(B) $2a_2\equiv a_1+a_3 \bmod 5$ *and* $2a_2\geqq a_1+a_3$,

(C) $2a_1\equiv a_2+a_3 \bmod 5$, $a_1+a_2\equiv 0 \bmod 5$ *and* $2a_3\geqq 2a_1+a_2$,

(D) $2a_1\equiv a_2+a_3 \bmod 5$, $a_1+a_3\equiv 0 \bmod 5$ *and* $2a_2\geqq 2a_1+a_3$.

*Then $H$ is Weierstrass.*

## §3. Numerical semigroups $H$ starting with 5 and satisfying $\#M(H)=5$.

In this section we will give sufficient conditions for a numerical semigroup $H$ starting with 5 and satisfying $\#M(H)=5$ to be Weierstrass.

**Lemma 3.1.** *Let $H$ be a numerical semigroup starting with $n$. Assume that $M(H)=\{a_0=n, a_1, \cdots, a_{n-1}\}$. $I_H$ denotes the kernel of the $k$-algebra homomorphism $\varphi_H: k[X_0, X_1, \cdots, X_{n-1}]\to k[t]$ defined by sending $X_i$ to $t^{a_i}$. For any $i$ and $j$ with $1\leqq i\leqq j\leqq n-1$, we set*

$$f_{ij}=X_iX_j-X_0^{e(i,j)}X_{r(i,j)}$$

*where $a_i+a_j=e(i, j)n+a_{r(i,j)}$. Then for all $i$ and $j$, we have $e(i, j)>0$. Moreover, the ideal $I_H$ is generated by $f_{ij}$'s $(1\leqq i\leqq j\leqq n-1)$.*

*Proof.* If $e(i, j)\leqq 0$, then this contradicts $a_{r(i,j)}\in M(H)$. Let $J$ be the ideal generated by $f_{ij}$'s. In view of $a_i+a_j=e(i, j)a_0+a_{r(i,j)}$, we get $J\subseteqq I_H$. Conversely we will show that $I_H\subseteqq J$. It is known that the ideal $I_H$ is generated by the element of type

$$\prod X_i^{\nu_i}-\prod X_i^{\mu_i},\ \nu_i\mu_i=0.$$

If $f=\prod X_i^{\nu_i}-\prod X_i^{\mu_i}\in I_H$ and $f\neq 0$, then we have $\sum \nu_i>1$ and $\sum \mu_i>1$, because of $a_i\in M(H)$. Moreover,

$$f=f_{jl}\prod X_i^{\nu_i'}-f_{pq}\prod X_i^{\mu_i'}+X_0^{e(j,l)}X_{r(i,j)}\prod X_i^{\nu_i'}-X_0^{e(p,q)}X_{r(p,q)}\prod X_i^{\mu_i'}$$
$$\equiv X_0^{e(j,l)}X_{r(i,j)}\prod X_i^{\nu_i'}-X_0^{e(p,q)}\prod X_i^{\mu_i'} \bmod J.$$

Hence we may decrease the weight of $f$, because $e(i, j)>0$, $e(p, q)>0$, $J\subseteqq I_H$ and the ideal $I_H$ is prime. Therefore, at last we get $f\in J$. *Q.E.D.*

Let $M(H)=\{a_0=5, a_1, a_2, a_3, a_4\}$. First in certain cases we will give minimal relations among $a_0$, $a_1$, $a_2$, $a_3$ and $a_4$.

**Lemma 3.2.** *Let* $M(H)=\{a_0=5,\ a_1=5q_1+1,\ a_2=5q_2+2,\ a_3=5q_3+3,\ a_4=5q_4+4\}$. *Suppose that* $2q_2-q_1-q_3\geqq 0$ *and* $2q_3-q_2-q_4\geqq 0$. *We set*

$$\alpha_i=\mathrm{Min}\{\alpha\in \boldsymbol{N}\geqq 2 \mid \alpha a_i\in\langle a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_4\rangle\}.$$

*Then we have* $\alpha_0=q_1+q_4+1$ *and* $\alpha_1=\alpha_2=\alpha_3=\alpha_4=2$. *In fact,*

(0, 0) $\quad \alpha_0 a_0=\alpha_{01}a_1+\alpha_{04}a_4,\ \alpha_{01}=\alpha_{04}=1,$

(1, 1) $\quad \alpha_1 a_1=\alpha_{10}a_0+\alpha_{12}a_2,\ \alpha_{10}=2q_1-q_2,\ \alpha_{12}=1,$

(2, 2) $\quad \alpha_2 a_2=\alpha_{20}a_0+\alpha_{21}a_1+\alpha_{23}a_3,\ \alpha_{20}=2q_2-q_1-q_3,\ \alpha_{21}=\alpha_{23}=1,$

(3, 3) $\quad \alpha_3 a_3=\alpha_{30}a_0+\alpha_{32}a_2+\alpha_{34}a_4,\ \alpha_{30}=2q_3-q_2-q_4,\ \alpha_{32}=\alpha_{34}=1,$

(4, 4) $\quad \alpha_4 a_4=\alpha_{40}a_0+\alpha_{43}a_3,\ \alpha_{40}=2q_4+1-q_3,\ \alpha_{43}=1.$

Moreover,

(1, 2) $\quad \alpha_{01}a_1+\alpha_{32}a_2=(\alpha_{10}+\alpha_{20})a_0+\alpha_{23}a_3,$

(1, 3) $\quad \alpha_{01}a_1+\alpha_{43}a_3=(\alpha_{10}+\alpha_{20}+\alpha_{30})a_0+\alpha_{34}a_4,$

(2, 3) $\quad \alpha_{12}a_2+\alpha_{43}a_3=(\alpha_{20}+\alpha_{30})a_0+\alpha_{21}a_1+\alpha_{34}a_4,$

(2, 4) $\quad \alpha_{12}a_2+\alpha_{04}a_4=(\alpha_{20}+\alpha_{30}+\alpha_{40})a_0+\alpha_{21}a_1,$

(3, 4) $\quad \alpha_{23}a_3+\alpha_{04}a_4=(\alpha_{30}+\alpha_{40})a_0+\alpha_{32}a_2.$

*Proof.* We have

$$(q_2+q_3+1)-(q_1+q_4+1)=2q_2-q_1-q_3+2q_3-q_2-q_4\geqq 0.$$

Hence $\alpha_0=q_1+q_4+1$. Computations show the remaining part. *Q.E.D.*

Applying Lemma 3.1 to our case we get the following:

**Propositon 3.3** *Let the notation be as in Lemma 3.2. Suppose that* $2q_2-q_1-q_3>0$ *and* $2q_3-q_2-q_4>0$. *We set*

$$g_1=X_0^{\alpha_{10}},\ g_2=X_0^{\alpha_{20}},\ g_3=X_0^{\alpha_{30}},\ g_4=X_0^{\alpha_{40}},\ g_5=X_1^{\alpha_{01}},\ g_6=X_1^{\alpha_{21}},$$
$$g_7=X_2^{\alpha_{32}},\ g_8=X_3^{\alpha_{43}},\ g_9=X_4^{\alpha_{04}},\ g_{10}=X_2^{\alpha_{12}},\ g_{11}=X_3^{\alpha_{23}},\ g_{12}=X_4^{\alpha_{34}}.$$

*Let S be the subsemigroup of* $\boldsymbol{Z}^8$ *generated by*

$$b_i=e_i(1\leqq i\leqq 8),\ b_9=e_1+e_2+e_3+e_4-e_5,\ b_{10}=-e_1+e_5+e_6,$$
$$b_{11}=-e_1-e_2+e_5+e_7 \text{ and } b_{12}=-e_1-e_2-e_3+e_5+e_8.$$

*where* $\boldsymbol{Z}^8$ *is the set of row vectors of dimension 8 with integral coefficients and for each* $i$, $e_i$ *denotes the row vector with i-th coefficient 1 and other coefficients 0. Then we get* $I_H=\eta(\mathrm{Ker}\pi)k[X]$ *where*

$$\pi:\ k[Y]=k[Y_1,\ \cdots,\ Y_{12}]\to k[T^s]_{s\in S}$$
$$(resp.\ \eta:\ k[Y]\to k[X]=k[X_0,\ X_1,\ \cdots,\ X_4])$$

*denotes the* $k$*-algebra homomorphism defined by* $\pi(Y_i)=T^{b_i}$ $(resp.\ \eta(Y_i)=g_i)$. *Moreover* $S$ *is saturated.*

*Proof.* Let $\langle\ ,\ \rangle:\ \boldsymbol{Z}^8\times\boldsymbol{Z}^8\to\boldsymbol{Z}$ be the pairing defined by $\langle r,\ a\rangle={r}^t a$ for any $r$ and $a\in\boldsymbol{Z}^8$. We define the map

$$\Psi_H:\ k[T^{b_i}]_{1\leqq i\leqq 12}\to k[H]$$

by sending $T^{b_i}$ to $t^{\langle\lambda,b_i\rangle}$ where

$$\lambda=(\alpha_{10}a_0,\ \alpha_{20}a_0,\ \alpha_{30}a_0,\ \alpha_{40}a_0,\ \alpha_{01}a_1,\ \alpha_{21}a_1,\ \alpha_{32}a_2,\ \alpha_{43}a_3).$$

Then we have $\varphi_H\circ\eta=\psi_H\circ\pi$, which implies that $\eta(\mathrm{Ker}\pi)\subseteqq\mathrm{Ker}\varphi_H=I_H$. Conversely we will show that $I_H\subseteqq\eta(\mathrm{Ker}\pi)k[X]$. By Lemma 3.1 the ideal $I_H$ is generated by $f_{00}$, $f_{11}$, $f_{22}$, $f_{33}$, $f_{44}$, $f_{12}$, $f_{13}$, $f_{23}$, $f_{24}$ and $f_{34}$ where $f_{ij}$'s are polynomials in $k[X]$ associated to the relation $(i,\ j)$ in Lemma 3.2, e.g. $f_{00}=X_0^{\alpha_0}-X_1^{\alpha_{01}}X_4^{\alpha_{04}}$. In fact,

$$a_2+a_2=(2q_2-q_4)a_0+a_4,\ a_3+a_3=(2q_3+1-q_1)a_0+a_1 \text{ and } a_2+a_3=(q_2+q_3+1)a_0.$$

Then we see $\quad X_2^2-X_0^{2q_2-q_4}X_4=f_{22}+X_0^{2q_2-q_1-q_3}f_{13},\ X_3^2-X_0^{2q_3+1-q_1}X_1=f_{33}+X_0^{2q_3-q_2-q_4}f_{24}$

and $$X_2X_3-X_0^{q_2+q_3+1}=f_{23}-X_0^{q_2+q_3-q_1-q_4}f_{00}.$$

Moreover, the following polynomials in $k[Y]$ are contained in $\mathrm{Ker}\pi$ :

$$Y_1Y_2Y_3Y_4-Y_5Y_9,\ Y_5Y_6-Y_1Y_{10},\ Y_7Y_{10}-Y_2Y_6Y_{11},$$
$$Y_8Y_{11}-Y_3Y_7Y_{12},\ Y_9Y_{12}-Y_4Y_8,\ Y_5Y_7-Y_1Y_2Y_{11},\ Y_5Y_8-Y_1Y_2Y_3Y_{12},$$
$$Y_{10}Y_8-Y_6Y_{12}Y_2Y_3,\ Y_{10}Y_9-Y_2Y_3Y_4Y_6 \text{ and } Y_{11}Y_9-Y_3Y_4Y_7.$$

By calculation we have $I_H\subseteqq\eta(\mathrm{Ker}\pi)k[X]$. Hence we get $I_H=\eta(\mathrm{Ker}\pi)k[X]$.

Lastly we will show that $S$ is saturated. It suffices to show that

$$\sum_{i=1}^{12}\boldsymbol{R}_+b_i\cap\boldsymbol{Z}^8=\sum_{i=1}^{12}\boldsymbol{N}b_i.$$

Let us take $y=\sum_{i=1}^{12}s_ib_i\in\boldsymbol{Z}^8$ with $s_i\in\boldsymbol{R}_+$. Then we may assume $0\leqq s_i<1$ for all $i$. Now

$$y=(s_1+s_9-s_{10}-s_{11}-s_{12},\ s_2+s_9-s_{11}-s_{12},\ s_3+s_9-s_{12},\ s_4+s_9,$$
$$s_5-s_9+s_{10}+s_{11}+s_{12},\ s_6+s_{10},\ s_7+s_{11},\ s_8+s_{12})\in\boldsymbol{Z}^8.$$

In the case $s_1+s_9-s_{10}-s_{11}-s_{12}=-2$ we have

$$s_2+s_9-s_{11}-s_{12}=-2+s_{10}-s_1+s_2<0,$$

which implies that $s_2+s_9-s_{11}-s_{12}=-1$. Hence we may assume that

$$y=(-2,\ -1,\ 0,\ 0,\ 2,\ 1,\ 1,\ 1).$$

Now

$$y=b_3+b_7+b_{10}+b_{12}\in\sum_{i=1}^{12}\boldsymbol{N}b_i.$$

In the case $s_1+s_9-s_{10}-s_{11}-s_{12}=-1$ we have

$$s_5-s_9+s_{10}+s_{11}+s_{12}=s_5+s_1+1.$$

Moreover, $s_{10}s_{11}\neq 0$ or $s_{12}\neq 0$. Hence we may assume that

$$y=(-1,\ -1,\ 0,\ 0,\ 1,\ 1,\ 1,\ 0)=b_6+b_{11}$$

or

$$y=(-1,\ -1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 1)=b_3+b_{12}.$$

Therefore we get $y\in\sum_{i=1}^{12}\boldsymbol{N}b_i$. The case of $s_1+s_9-s_{10}-s_{11}-s_{12}\geqq 0$ and $s_2+s_9-s_{11}-s_{12}=-1$ does not occur. Hence we get our desired result. *Q.E.D.*

In the case of $2q_2-q_1-q_3=0$ or $2q_3-q_2-q_4=0$ we also obtain the result which is similar to Proposion 3.3. Hence we get

**Theorem 3.4.** *Let $H$ be a numerical semigroup with*

$$M(H)=\{a_0=5,\ a_1=5q_1+1,\ a_2=5q_2+2,\ a_3=5q_3+3,\ a_4=5q_4+4\}.$$

*If $2q_2-q_1-q_3\geqq 0$ and $2q_3-q_2-q_4\geqq 0$, then $H$ is Weierstrass*[2].

In the similar way we get the following:

**Theorem 3.5** *Let $H$ be a numerical semigroup with*

$$M(H)=\{a_0=5,\ a_1=5q_1+1,\ a_2=5q_2+2,\ a_3=5q_3+3,\ a_4=5q_4+4\}.$$

*If $2q_1-q_3-q_4-1\geqq 0$ and $2q_4+1-q_1-q_2\geqq 0$, then $H$ is Weierstrass.*

*Proof.* Replacing suffixes 1, 2, 3 and 4 in the proof of Theorem 3.4 by 3, 1, 4 and 2 respectively, we get the proof of Theorem 3.5 *Q.E.D.*

Lastly we show that the condition of Theorem 3.4 and that of Theorem 3.5 are disjoint.

**Remark 3.6.** Let

$$I=\{(q_1,\ q_2,\ q_3,\ q_4)\in\boldsymbol{N}^4\mid 2q_2-q_1-q_3\geqq 0,\ 2q_3-q_2-q_4\geqq 0\}$$

and

$$J=\{(q_1,\ q_2,\ q_3,\ q_4)\in\boldsymbol{N}^4\mid 2q_1-q_3-q_4-1\geqq 0,\ 2q_4+1-q_1-q_2\geqq 0\}.$$

Then we have $I\cap J=\phi$.

*Proof.* Let $(q_1,\ q_2,\ q_3,\ q_4)\in I\cap J$. Since

$$(2q_2-q_1-q_3)+(2q_3-q_2-q_4)=-((2q_1-q_3-q_4-1)+(2q_4+1-q_1-q_2)),$$

$(q_1, q_2, q_3, q_4)$ satisfies the simultaneous linear equatuions

$$2q_1-q_3-q_4-1=0,\ 2q_2-q_1-q_3=0,\ 2q_3-q_2-q_4=0,\ 2q_4+1-q_1-q_2=0.$$

Then the solutions of the above are

$$(q_1, q_2, q_3, q_4)=\left(\frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0\right)+\boldsymbol{R}(1, 1, 1, 1).$$

Since $q_i$'s must be integers, this is a contradiction. Hence we obtain $I\cap J=\phi$. *Q.E.D.*

## §4. Numerical semigroups of genus $g\leqq 10$ stating with 5.

In the last section applying our results we will investigate whether a numerical semigroup of genus $g\leqq 10$ starting with 5 is Weierstrass.

**Example 4.1.** Any numerical semigroup of genus $g\leqq 6$ is Weierstrass.

*Proof.* It suffices to show that all numerical semigroups starting with $a\geqq 5$ are Weierstrass[1),2)]. All numerical semigroups of genus 3 start with $a\leqq 4$. Any numerical semigroup of genus 4 except $\langle 5, 6, 7, 8, 9\rangle$ starts with $a\leqq 4$. Numerical semigroups of genus 5 starting with $a\geqq 5$ are the following:

5.1) $\langle 6, 7, 8, 9, 10, 11\rangle$ 5.2) $\langle 5, 7, 8, 9, 11\rangle$ 5.3) $\langle 5, 6, 8, 9\rangle$
5.4) $\langle 5, 6, 7, 9\rangle$ 5.5) $\langle 5, 6, 7, 8\rangle$.

Now 5.1) is the set of non-gaps at an ordinary point. 5.2), 5.3), 5.4) and 5.5) are negatively graded, which implies that they are Weierstrass[6),7)]. In the similar way any numerical semigroup of genus 6 starting with $a\geqq 6$ is Weierstrass. Numerical semigroups of genus 6 starting with 5 are the following:

6.1) $\langle 5, 8, 9, 11, 12\rangle$ 6.2) $\langle 5, 7, 9, 11, 13\rangle$ 6.3) $\langle 5, 7, 8, 11\rangle$
6.4) $\langle 5, 7, 8, 9\rangle$ 6.5) $\langle 5, 6, 9, 13\rangle$ 6.6) $\langle 5, 6, 8\rangle$ 6.7) $\langle 5, 6, 7\rangle$.

By Proposition 1.5, 6.1) and 6.2) are 5-cyclic, hence Weierstrass. By Theorem 2.2, 6.3), 6.4) and 6.5) are Weierstrass. Moreover, 6.6) and 6.7) are generated by three elements, which implies that they are Weierstrass[3)]. *Q.E.D.*

**Example 4.2.** Any numerical semigroup of genus 7 at least except $\langle 5, 7, 11, 13\rangle$ is Weierstrass. I don't know whether $\langle 5, 7, 11, 13\rangle$ is Weierstrass or not.

*Proof.* Numerical semigroups of genus 7 starting with $a\geqq 7$ are negatively graded, which implies that they are Weierstrass. For any numerical semigroup $H$ we denote by $w(H)$ the weight of $H$, i.e. $w(H)=\sum_{i=1}^{g}(n_i-1)$ where $g=g(H)$ and $n_1, \cdots, n_g$ are the elements of $\boldsymbol{N}-H$. Then we note the following result: if a numerical semigroup $H$ starting with $a$ satisfies $w(H)<a$, then $H$ is Weierstrass[8)]. Numerical semigroups $H$ of

genus 7 starting with 6 which satisfy $w(H)\geqq 6$ are the following:

7.1) $\langle 6, 7, 8, 9\rangle$ 7.2) $\langle 6, 7, 8, 10\rangle$ 7.3) $\langle 6, 7, 8, 11\rangle$ 7.4) $\langle 6, 7, 9, 10\rangle$

First we consider the case 7.1). Let $C$ be a curve defined by an equation of the form

$$y^6=(x-c_1)(x-c_2)(X-c_3)(x-c_4)^4$$

where $c_i$'s are distinct elements of $k$. Let $f: C\rightarrow \boldsymbol{P}^1$ be the surjective morphism of degree 6 corresponding to the inclusion $k(x)\subset k(x, y)=K(C)$ where $K(C)$ is the function field of $C$. If $\{P_\infty\}=f^{-1}((0, 1))$, then we have $H(P_\infty)=\langle 6, 7, 8, 9\rangle$. In the case 7.2) if we set $H=\langle 6, 7, 9, 10\rangle$, then the ideal $I_H$ is generated by

$$X_0^3-X_2X_3,\ X_1^2-X_0X_2,\ X_2^2-X_0X_3 \text{ and } X_3^2-X_0X_1^2.$$

Hence $H$ is Weierstrass[3]. In the case 7.3) the semigroup is 1-neat, hence Weierstrass[5]. Lastly we consider the case 7.4). Let $E$ be an elliptic curve with the origin $Q'$ and let $P_2'$, $P_3'$ be two distinct points of $E$ such that $P_2'+P_3'\neq Q'$. Let $P_1'=-2P_2'-2P_3'$. Take $z\in K(E)$ such that $\operatorname{div}(z)=P_1'+2P_2'+3P_3'-5Q'$. Let $\pi: C\rightarrow E$ be the surjective morphism corresponding to the includion $K(E)\subset K(E)(y)=K(C)$ with $y=z^{\frac{1}{5}}$. We set $\{P_i\}=\pi^{-1}(P_i')$ for $i=1, 2, 3$. Suppose that $Q'\neq P_i'$ for $i=1, 2, 3$ and that $3P_2'+3P_3'=Q'$. Then we have

$$\operatorname{div}_C(y)=P_1+2P_2+2P_3-\pi^*(Q'),$$
$$\operatorname{div}_C(dy)=-2\pi^*(Q')+P_2+P_3+\sum_{i=1}^{4}\pi^*(R_i')$$

where $R_1'$, $R_2'$, $R_3'$ and $R_4'$ are points of $E$ which are distinct from $P_1'$, $P_2'$, $P_3'$ and $Q'$. Moreover, for any $r\in\boldsymbol{N}$ and any $f\in K(E)$ with $\operatorname{div}_E(f)=\sum_{P\in E} n(P')P'$ we obtain

$$\operatorname{div}_C\left(\frac{fdy}{y^{1-r}}\right)=(5n(P_1')+(r-1))P_1+\sum_{i=2}^{3}(5n(P_i')+1+2(r-1))P_i$$
$$+(n(Q')-r-1)\pi^*(Q')+\sum_{i=1}^{4}(n(R_i')+1)\pi^*(R_1')+\Sigma' n(P')\pi^*(P')$$

where $\Sigma'$ means the summation over all $P'\in E$ except $P_1'$, $P_2'$, $P_3'$, $R_j$'s and $Q'$. For any divisor $D'$ on $E$ we set

$$L(D')=\{f\in K(E)\,|\,\operatorname{div}(f)\geqq -D'\} \text{ and } l(D')=\dim_k L(D').$$

Hence for any $r\in\{0, 1, 2, 3, 4\}$, $\dfrac{fdy}{y^{1-r}}$ has no poles if and only if $f\in L(D_r')$ where

$$D_0'=-P_1'-P_2'-P_3'-Q'+\sum_{i=1}^{4}R_i',\ D_1'=-2Q'+\sum_{i=1}^{4}R_i',\ D_2'=-3Q'+\sum_{i=1}^{4}R_i',$$
$$D_3'=-4Q'+P_2'+P_3'+\sum_{i=1}^{4}R_i' \text{ and } D_4'=-5Q'+P_2'+P_3'+\sum_{i=1}^{4}R_i'.$$

Moreover, we obtain

$$l(D_0')=1,\ l(D_0'-P_1')=0,\ l(D_1')=2,\ l(D_1'-P_1')=l(D_1'-2P_1')=1,$$
$$l(D_1'-3P_1')=0,\ l(D_2')=1,\ l(D_2'-P_1')=0,\ l(D_3')=2,\ l(D_3'-P_1')=1,$$
$$l(D_3'-2P_1')=0,\ l(D_4')=1,\ l(D_4'-P_1')=0,$$

which imply that $\boldsymbol{N}-H(P_1)=\{1, 2, 3, 4, 5, 8, 11\}$, hence $H(P_1)=\langle 6, 7, 9, 10\rangle$. If $H$ is a numerical semigroup of genus 7 starting with 5 which satisfies $\#M(H)\geqq 4$ and $w(H)\geqq 5$, then it is one of the following:

7.5) $\langle 5, 6, 13, 14\rangle$ 7.6) $\langle 5, 7, 9, 11\rangle$ 7.7) $\langle 5, 7, 9, 13\rangle$
7.8) $\langle 5, 7, 11, 13\rangle$ 7.9) $\langle 5, 8, 9, 11\rangle$ 7.10) $\langle 5, 8, 9, 12\rangle$.

In view of Theorem 2.2, 7.5), 7.6), 7.7) and 7.9) are Weierstrass. Moreover, 7.10) is Weierstrass[9]. *Q.E.D.*

**Example 4.3.** All numerical semigroups of genus 8 starting with 5 at least except $\langle 5, 9, 11, 12\rangle$ are Weierstrass.

*Proof.* Any numerical semigroup of genus 8 starting with 5 is one of the following:

$\langle 5, 11, 12, 13, 14\rangle$, $\langle 5, 9, 12, 13, 16\rangle$, $\langle 5, 9, 11, 13, 17\rangle$, $\langle 5, 9, 11, 12\rangle$, $\langle 5, 8, 12, 14\rangle$, $\langle 5, 8, 11, 14, 17\rangle$, $\langle 5, 8, 11, 12\rangle$, $\langle 5, 8, 9\rangle$, $\langle 5, 7, 13, 16\rangle$, $\langle 5, 7, 11\rangle$, $\langle 5, 7, 9\rangle$, $\langle 5, 6, 14\rangle$, $\langle 5, 6, 13\rangle$.

By Proposition 1.5 and Theorems 2.2, 3.4, 3.5, we get our result. *Q.E.D.*

**Example 4.4.** All numerical semigroups of genus 9 starting with 5 at least except $\langle 5, 11, 13, 14, 17\rangle$, $\langle 5, 11, 12, 14, 18\rangle$, $\langle 5, 11, 12, 13, 19\rangle$, $\langle 5, 9, 12, 16\rangle$ and $\langle 5, 8, 14, 17\rangle$ are Weierstrass.

*Proof.* Any numerical semigroup of genus 9 starting with 5 is one of the following:

$\langle 5, 12, 13, 14, 16\rangle$, $\langle 5, 11, 13, 14, 17\rangle$, $\langle 5, 11, 12, 14, 18\rangle$, $\langle 5, 11, 12, 13, 19\rangle$, $\langle 5, 9, 13, 16, 17\rangle$, $\langle 5, 9, 12, 16\rangle$, $\langle 5, 9, 12, 13\rangle$, $\langle 5, 9, 11, 17\rangle$, $\langle 5, 9, 11, 13\rangle$, $\langle 5, 8, 14, 17\rangle$, $\langle 5, 8, 12, 19\rangle$, $\langle 5, 8, 11, 17\rangle$, $\langle 5, 8, 11, 14\rangle$, $\langle 5, 7, 16, 18\rangle$, $\langle 5, 7, 13\rangle$, $\langle 5, 6, 19\rangle$.

Applying our results in sections 1, 2 and 3 to these cases we get the above statement. *Q.E.D.*

**Example 4.5.** All numerical semigroups of genus 10 starting with 5 at least except $\langle 5, 12, 13, 14, 21\rangle$ and $\langle 5, 11, 13, 14\rangle$ are are Weierstrass.

*Proof.* Any numerical semigroup of genus 10 starting with 5 is one of the following:

$\langle 5, 13, 14, 16, 17\rangle$, $\langle 5, 12, 14, 16, 18\rangle$, $\langle 5, 12, 13, 16, 19\rangle$, $\langle 5, 12, 13, 14, 21\rangle$, $\langle 5, 11, 14, 17, 18\rangle$, $\langle 5, 11, 13, 17, 19\rangle$, $\langle 5, 11, 13, 14\rangle$, $\langle 5, 11, 12, 18, 19\rangle$, $\langle 5, 11, 12, 14\rangle$, $\langle 5, 11, 12, 13\rangle$, $\langle 5, 9, 16, 17\rangle$, $\langle 5, 9, 13, 17, 21\rangle$, $\langle 5, 9, 13, 16\rangle$, $\langle 5, 9, 12\rangle$, $\langle 5, 9, 11\rangle$, $\langle 5, 7, 18\rangle$, $\langle 5, 7, 16\rangle$, $\langle 5, 6\rangle$, $\langle 5, 8, 17, 19\rangle$.

Applying our results in sections 1, 2 and 3 to these cases, we get the above statement. *Q.E.D.*

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