

# The Weierstrass gap Sequences of Certain Ramification Points of Tetragonal Coverings of $P^1$

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## Abstract

Let  $C$  be a curve, where a *curve* means a complete non-singular irreducible 1-dimensional algebraic variety over an algebraically closed field  $k$  of characteristic zero. We assume that  $C$  has a base-point free 1-dimensional tetragonal linear system  $g_{4d}^1$ .  $f: C \rightarrow P^1$  denotes a tetragonal covering associated to  $g_{4d}^1$ . In this paper we give the Weierstrass gap sequence of a ramification point of  $f$  whose index is 4 or 3.

## Notation

Throughout this paper we will use the following notation without further warning.  $N$  denotes the additive semigroup of non-negative integers. Let  $C$  be a curve of genus  $g$  and let  $D$  be a divisor on  $C$ . We write  $l(D)$  instead of  $\dim H^0(C, \mathcal{O}_C(D))$ . Moreover, we set  $i(D) = l(K - D)$  where  $K$  denotes a canonical divisor on  $C$ . Let  $P$  be a point of  $C$ . We say that  $n$  is a *gap* of  $P$  if  $l((n-1)P) = l(nP)$ . Otherwise, we say that  $n$  is a *non-gap* of  $P$ . Then one can prove that the number of gaps of  $P$  is equal to  $g$  and each one of them is at most equal to  $2g-1$ . Let  $1 \leq n_1 < n_2 < \dots < n_g < 2g$  be such that  $n_i$  is a gap of  $P$  for each  $1 \leq i \leq g$ . Then  $(n_1, n_2, \dots, n_g)$  is called the *Weierstrass gap sequence* of  $P$ .

## § 1. Tetragonal Linear Systems of Type $(n, m)$ .

In the first section we classify 1-dimensional tetragonal linear systems  $g_4^1$  on curves, and calculate  $l(rg_4^1)$ . Let  $C$  be a curve of genus  $g$  and let  $g_4^1$  be a 1-dimensional tetragonal linear system on  $C$ .

**Definition.** Let  $n$  and  $m$  be positive integers with  $n \leq m$ . Suppose that

$$l(ng_4^1) = n+1, \quad l((n+1)g_4^1) > l(ng_4^1)+1, \quad l(mg_4^1) \leq l((m-1)g_4^1)+2$$

and

$$l((m+1)g_4^1) > l(mg_4^1)+2.$$

Then we say that  $g_4^1$  is of *type*  $(n, m)$ .

By the theorem of Riemann-Roch, we get

$$n+1 = l(ng_4^1) = 4n+1-g + l(K-ng_4^1) \geq 4n+1-g,$$

which implies that  $n \leq g/3$ . Moreover, we see easily that

$$l(mg_4^1) = l((m-1)g_4^1) + 2 = \cdots = l(ng_4^1) + 2(m-n) = 2m - n + 1$$

and that

$$l(mg_4^1) = 4m+1-g + l(K-mg_4^1) \geq 4m+1-g$$

Hence we get  $m \leq (g-n)/2$ . In the same way as in the trigonal case<sup>1)</sup> we obtain the following.

**Lemma 1.** *Let  $k$  be a positive integer. Then  $l((k+1)g_4^1) = l(kg_4^1) + 4$  if and only if the divisor  $kg_4^1$  is not special.*

**Proposition 2.** *Let  $g_4^1$  be of type  $(n, m)$ . If  $k \in \mathbf{N}$  with  $k \leq m-n$ , then  $l((n+k)g_4^1) = n+1+2k$ . If  $k \in \mathbf{N}$  such that  $k > m-n$  and  $(n+k-1)g_4^1$  is special, then  $l((n+k)g_4^1) = 2n - m + 3k + 1$ .*

*Proof.* If  $n+k \leq m$ , then

$$l((n+k)g_4^1) = l(ng_4^1) + 2k = n+1+2k.$$

Hence  $l(mg_4^1) = n+1+2(m-n)$ . If  $(n+k-1)g_4^1$  is special, by Lemma 1

$$l((n+k)g_4^1) \leq l((n+k-1)g_4^1) + 3.$$

Hence for  $m < n+k$  we have

$$l((n+k)g_4^1) = l(mg_4^1) + 3(k+n-m) = n+1+2(m-n) + 3(k+n-m) = 2n - m + 3k + 1. \text{ Q.E.D.}$$

**Proposition 3.** *Let  $g_4^1$  be of type  $(n, m)$  and let  $k \in \mathbf{N}$ . The linear system  $(n+k)g_4^1$  is special if and only if  $k \leq g-2n-m-1$ .*

*Proof.*  $(n+k)g_4^1$  is special if and only if  $l((n+k)g_4^1) > 4n+4k+1-g$ . Using Proposition 2 we get  $k \leq g-2n-m-1$ . Q.E.D.

## § 2. The Weierstrass Gap Sequence of a Total Ramification Point.

Let  $g_4^1$  be a base-point free 1-dimensional tetragonal linear system of type  $(n, m)$  on a curve  $C$  of genus  $g$  and let  $f: C \rightarrow \mathbf{P}^1$  be the morphism associated to  $g_4^1$ . Then using results in §1 we obtain the Weierstrass gap sequence of a total ramification point  $P$  of  $f$ , i.e. the ramification index of  $f$  at  $P$  is equal to 4.

**Proposition 4.** *The gaps of  $P$  are given in the list below. For any  $0 \leq i < n$ ,  $4i+1, 4i$*

$+2, 4i+3$ , and for any  $n \leq i < m, 4i+u, 4i+v$ , and for any  $m \leq i < g-n-m, 4i+u$  where  $u, v \in \{1, 2, 3\}$  with  $u \neq v$ .

*Proof.* Since  $P$  is a total ramification point of  $f$ , we have  $4P \in g_4^1$ , hence 4 is the first non-gap of  $P$ . Because  $g_4^1$  is of type  $(n, m)$ , we see that

$$n+1 = l(4nP), n+2 < l(4(n+1)P), l(4(m-1)P)+2 \geq l(4mP)$$

and

$$l(4mP)+2 < l(4(m+1)P).$$

By Proposition 3 we have  $i(4(g-n-m-1)P) \geq 1$  and  $i(4(g-n-m)P) = 0$ . Hence by the Riemann-Roch theorem we get our desired result. Q.E.D.

It is proved that the above linear system always exists. In fact, for any subsemigroup  $H$  of  $N$  whose complement in  $N$  is finite and whose least positive element is 4, there exists a curve  $C$  such that  $H = H(P)$  for some  $P \in C$  where  $H(P)$  denotes the subsemigroup of  $N$  whose elements are the non-gaps of  $P$ <sup>2)</sup>. Hence we obtain the following:

**Proposition 5.** *Let  $g, n$  and  $m$  be positive integers satisfying  $n \leq m \leq (g-n)/2$ . Let  $G$  be the set whose elements are the gaps listed in Proposition 4. Assume that  $N-G$  forms a subsemigroup of  $N$ . Then there exists a base-point free 1-dimensional tetragonal linear system  $g_4^1$  of type  $(n, m)$  on some curve  $C$  of genus  $g$  with  $4P \in g_4^1$  and  $H(P) = N-G$  for some  $P \in C$ .*

### § 3. The Weierstrass Gap Sequence of a Ramification Point whose Index is 3

Let  $g_4^1$  be a base-point free 1-dimensional tetragonal linear system of type  $(n, m)$  on a curve of genus  $g$  and let  $f: C \rightarrow P^1$  be the morphism associated to  $g_4^1$ . Let  $P$  be a ramification point of  $f$  whose index is 3. In this section the Weierstrass gap sequence of  $P$  is given.

**Lemma 6.** *The first non-gap of  $P$  is larger than  $3n$ .*

*Proof.* Assume that  $l(3nP) > 1$ . Let  $Q$  be a point of  $C$  with  $Q \neq P$  such that  $3P+Q \in g_4^1$ . In view of

$$l(3nP+nQ) = l(ng_4^1) = n+1,$$

we see that  $l(3nP+iQ) = l(3nP+(i-1)Q)$  for some  $1 \leq i \leq n$ , hence  $Q$  is a fixed point of the linear system  $|3nP+iQ|$ . But

$$3nP+iQ \in ig_4^1+3(n-i)P,$$

hence we obtain a contradiction. Therefore, we get  $l(3nP)=1$ .

*Q.E.D.*

The above Lemma implies the following.

**Proposition 7.** *If  $n=g/3$ , then  $P$  is not a Weierstrass point.*

**Lemma 8.** (1) *For any integer  $b$  with  $n \leq b \leq m-1$ ,  $P$  has exactly  $b-n+1$  non-gaps between  $3n$  and  $3b+3$ .*

(2) *For any integer  $b$  with  $m \leq b \leq g-n-m-1$ ,  $P$  has exactly  $2b-m-n+2$  non-gaps between  $3n$  and  $3b+3$ .*

*Proof.* Let  $Q$  be as in the proof of Lemma 6. Let  $b$  be an integer with  $n \leq b \leq m-1$ . It follows from Proposition 2 that  $l((3b+3)P+(b+1)Q)=2b-n+3$ . Hence we get  $l((3b+3)P) \geq b-n+2$ . Assume that  $l((3b+3)P) > b-n+2$ . Then there exists  $1 \leq i \leq b+1$  such that  $l((3b+3)P+iQ) = l((3b+3)P+(i-1)Q)$ . Hence  $Q$  is a fixed point of  $ig_4^1+3(b+1-i)P$ , which is a contradiction. Therefore we obtain  $l((3b+3)P)=b-n+2$ . It follows from Lemma 6 that  $P$  has exactly  $b-n+1$  non-gaps between  $3n$  and  $3b+3$ . Let  $b$  be an integer with  $m \leq b \leq g-n-m-1$ . Proposition 2 implies that  $l((3b+3)P+(b+1)Q)=3b-m-n+4$ . As in the foregoing proof, we obtain (2). *Q.E.D.*

**Lemma 9.** *Let  $a$  be a positive integer with  $a \leq 3(g-n-m-1)$ . If  $a$  and  $a+1$  are non-gaps of  $P$ , then  $a+2$  is a gap of  $P$ .*

*Proof.* Let  $Q$  be as in the proof of Lemma 6. In view of Proposition 3,  $(a-1)P$  is a special divisor. Since  $a$  and  $a+1$  are non-gaps of  $P$ , we have  $i((a+1)P)=i(aP)=i((a-1)P)$ . We see that  $(a+2)P+Q \in (a-1)P+g_4^1$ . Hence  $Q$  is not a fixed point of the linear system  $| (a+2)P+Q |$ . Assume that  $a+2$  is a non-gap of  $P$ . Then we obtain

$$l(K-(a-1)P-g_4^1)=i((a+2)P+Q)=i((a+2)P)=i((a+1)P)=i((a-1)P)=l(K-(a-1)P).$$

Therefore  $g_4^1$  is a fixed part of  $|K-(a-1)P|$ , which is a contradiction. Hence  $a+2$  is a gap of  $P$ . *Q.E.D.*

**Proposition 10.** *If  $a$  is a non-gap of  $P$ , then so is  $a+3$ .*

*Proof.* We may assume that  $a \leq 3r$  where we set  $r=g-n-m-1$ . In fact, by Lemma 8 (2)  $P$  has exactly  $2g-3n-3m$  non-gaps between  $3n$  and  $3r+3$ . Hence  $P$  has exactly  $g$  gaps between 1 and  $3r+3$ . It follows that for any  $a > 3r$ ,  $a+3$  is a non-gap of  $P$ . In the remainder of this proof we assume that  $a \leq 3r$ .

First we assume that  $a \equiv 1 \pmod{3}$ , i.e.  $a=3b+1$ . Let  $n \leq b < m$ . Because of Lemma 8 (1) there are  $b-n$  non-gaps between  $3n$  and  $3b$ , and  $b-n+1$  non-gaps between  $3n$  and  $3b+3$ . Hence  $a$  is the  $(b-n+1)$ -th non-gap of  $P$ , which implies that  $l(aP)=b-n+2$  and  $l((a$

$-1)P = b - n + 1$ . On the other hand, by Proposition 2 we have  $l((a-1)P + bQ) = l(bg_4^1) = 2b - n + 1$ . Hence it follows that  $Q$  is a fixed point of  $|K - (a-1)P|$ . Let  $D' \in |K - (a+2)P|$ . We may assume that this divisor exists, because  $|K - (a+2)P| = \emptyset$  implies that  $a+3$  is a non-gap of  $P$ . We have  $D' + 3P \in |K - (a-1)P|$ , hence  $D' = D'' + Q$  for some effective divisor  $D''$  because  $Q$  is a fixed point of  $|K - (a-1)P|$ . Let  $\tilde{D} \in g_4^1$  with  $P \notin \text{Supp}(\tilde{D})$ , then it follows that  $\tilde{D} + D'' \in |K - (a-1)P|$ . But  $P$  is a fixed point of  $|K - (a-1)P|$ , hence  $P \in \text{Supp}(D'')$  and therefore  $P \in \text{Supp}(D')$ . It follows that  $P$  is a fixed point of  $|K - (a+2)P|$ , which means that  $a+3$  is a non-gap of  $P$ . Let  $m \leq b < r$ . In view of Lemma 8 (2) there are  $2b - m - n$  non-gaps between  $3n$  and  $3b$ , and  $2b - m - n + 2$  non-gaps between  $3n$  and  $3b + 3$ . Hence  $a$  is the  $(2b - m - n + 1)$ -th non-gap of  $P$ , which implies that  $l(aP) = 2b - m - n + 2$  and  $l((a-1)P) = 2b - m - n + 1$ . On the other hand, by Proposition 2 we have  $l((a-1)P + bQ) = l(bg_4^1) = 3b - n - m + 1$ . As in the above proof we see that  $a+3$  is a non-gap of  $P$ .

Secondly we assume that  $a \equiv 2 \pmod{3}$ , i.e.  $a = 3b + 2$ . Let  $n \leq b < m$ . Using Lemma 8, we see that  $l(aP) = b - n + 2$  and  $l((a-1)P) = b - n + 1$ . It follows from Proposition 2 that  $l((a-2)P + bQ) = l(bg_4^1) = 2b - n + 1$ . As in the foregoing proof we see that  $a+3$  is a non-gap of  $P$ . Let  $m \leq b < r$ . Because of Lemma 8 (2) there are  $2b - m - n$  non-gaps between  $3n$  and  $3b = a - 2$ , and  $2b - m - n + 2$  non-gaps between  $3n$  and  $3b + 3 = a + 1$ . If  $a - 1$  is a gap of  $P$ , then  $a$  is the  $(2b - m - n + 1)$ -th non-gap of  $P$ , which implies that  $l(aP) = 2b - m - n + 2$  and  $l((a-1)P) = 2b - m - n + 1$ . Moreover, we have  $l((a-2)P + Q) = 3b - n - m + 1$ . As in the foregoing proof, we see that  $a+3$  is a non-gap of  $P$ . Assume that  $a - 1$  is a non-gap of  $P$ . Then  $a$  is the  $(2b - m - n + 2)$ -th non-gap of  $P$ , which implies that  $l(aP) = 2b - m - n + 3$  and  $l((a-1)P) = 2b - m - n + 2$ . Moreover, we have

$$l((a-2)P + bQ) = 3b - n - m + 1 = l((a-1)P) + b - 1 = 3b + 1 + 1 - g + l(K - (a-1)P) + b - 1$$

and

$$l((a-2)P + bQ) = 3b + 1 - g + l(K - (a-2)P - bQ) + b.$$

Hence we get

$$l(K - (a-2)P - bQ) = l(K - (a-1)P) = i((a-1)P) = i((a-2)P) = l(K - (a-2)P),$$

which implies that  $Q$  is a fixed point of  $|K - (a-2)P|$ . As in the foregoing proof we see that  $a+3$  is a non-gap of  $P$ .

Lastly we assume that  $a \equiv 0 \pmod{3}$ , i.e.  $a = 3b$ . Let  $n \leq b < m - 1$ . Then there are  $b - n$  non-gaps between  $3n$  and  $a$ . Hence we have  $l(aP) = b - n + 1$ . Moreover, we see that  $l(aP + bQ) = 2b - n + 1 = l(aP) + b$ . On the other hand we have

$$l(aP + bQ) = 3b + 1 - g + l(K - aP - bQ) + b = l(aP) - l(K - aP) + l(K - aP - bQ) + b,$$

which implies that  $l(K - aP) = l(K - aP - bQ)$ . Hence  $Q$  is a fixed point of  $|K - aP|$ . As in the foregoing proof we see that  $a+3$  is a non-gap of  $P$ . Let  $m \leq b < r$ . Then there are  $2b - m - n$  non-gaps between  $3n$  and  $a$ , and  $2b - m - n + 2$  non-gaps between  $3n$  and  $a + 3$ . If

$a+1$  is a gap of  $P$ , then  $a+3$  is a non-gap of  $P$ . Hence we assume that  $a+1$  is a non-gap of  $P$ . Because of Lemma 9  $a+2$  is a gap of  $P$ , which implies that  $a+3$  is a non-gap of  $P$ . Q.E.D.

Combining Proposition 10 with Lemmas 6 and 8, we obtain the following main theorem in this paper.

**Theorem 11.** *The gaps of  $P$  are given in the list below. For any  $0 \leq i < n$ ,  $3i+1$ ,  $3i+2$ ,  $3i+3$ , and for any  $n \leq i < m$ ,  $3i+u$ ,  $3i+v$ , and for any  $m \leq i < g-n-m$ ,  $3i+u$  where  $u, v \in \{1, 2, 3\}$ , with  $u \neq v$ .*

**Remark 12.** Coppens has determined the Weierstrass gap sequences of the ramification points of the trigonal coverings of  $\mathbf{P}^1$  [1,3]. We have modified his method and applied it to our case.

#### § 4. On the Ramification Points whose Indices are Two

In these cases we obtain few results. In fact, applying Lewittes' theorem to some Galois cases we see that the ramification points are Weierstrass points.

**Proposition 13.** *Let  $C$  be a non-hyperelliptic curve of genus  $g \geq 4$  with an automorphism  $T$  such that the quotient curve  $C/\langle T \rangle$  is isomorphic to  $\mathbf{P}^1$ . Then any ramification point of the canonical morphism  $f: C \rightarrow C/\langle T \rangle \cong \mathbf{P}^1$  is a Weierstrass point.*

*Proof.* By the assumption,  $C$  is defined by an equation of the form

$$z^4 = \prod_{j=1}^{i_1} (x - c_j) \prod_{j=1}^{i_2} (x - c_{i_1+j})^2 \prod_{j=1}^{i_3} (x - c_{i_1+i_2+j})^3$$

where  $c'_j$ 's ( $1 \leq j \leq i_1 + i_2 + i_3$ ) are distinct elements of  $k$  and  $i_1 + i_3$  is odd, and the morphism  $f$  is induced by the inclusion  $\mathbf{K}(\mathbf{P}^1) = k(x) \subset k(x, z) = \mathbf{K}(C)$ , where for any curve  $X$ ,  $\mathbf{K}(X)$  denotes the function field of  $X$ . The indices of the ramification points of  $f$  are two or four. The ramification point whose index is four is a Weierstrass point. Let  $P$  be a ramification point whose index is two. Then  $P$  is a fixed point of  $T^2$ . We set

$$y = z^2 / \prod_{j=1}^{i_2+i_3} (x - c_{i_1+j}).$$

Then we have  $\mathbf{K}(C/\langle T^2 \rangle) = k(x, y)$  and

$$y^2 = \prod_{j=1}^{i_1} (x - c_j) \prod_{j=1}^{i_3} (x - c_{i_1+i_2+j}).$$

Hence we see that the genus of  $C/\langle T^2 \rangle$  is  $r$  where  $i_1 + i_3 = 2r + 1$ . Then the number of the fixed points of  $T^2$  is equal to  $2r + 2 + 2i_2$ , because the genus  $g$  of  $C$  is equal to  $3r + i_2$ . Because of  $r \geq 1$  and  $g \geq 4$ , the number of the fixed points of  $T^2$  is larger than 4. By Lewittes'

theorem<sup>4)</sup> any fixed point of  $T^2$ , i.e. any ramification point of  $f$  whose index is two, is a Weierstrass point. Q.E.D.

**Proposition 14.** *Let  $C$  be a curve of genus  $g \geq 4$  which is neither hyperelliptic nor elliptic-hyperelliptic. Assume that there exists a subgroup  $G$  of  $(2, 2)$ -type of the automorphism group of  $C$  with  $C/G \cong P^1$ . Let  $G_1, G_2$  and  $G_3$  be the subgroups of  $G$  of order 2. For any  $1 \leq i \leq 3$ , let  $g_i$  be the genus of  $C/G_i$ . Assume that  $g_1 \leq g_2 \leq g_3$ . If  $g_3 \neq g_1 + g_2$  and  $g_3 \neq g_1 + g_2 - 1$ , then any ramification point of the canonical morphism  $f : C \rightarrow C/G$  is a Weierstrass point.*

*Proof.* We see easily that  $g_3 = g_2 + g_1 - n + 1$  for some  $0 \leq n \leq g_1$ . Let  $r_i$  be the number of the ramification points of the canonical morphism  $f_i : C \rightarrow C/G_i$ , i.e. the fixed points of the automorphism  $T_i$  where  $G_i = \langle T_i \rangle$ . Because of Accola's theorem<sup>4)</sup> we have

$$r_3 = 2g - 2 - 4g_3 + 4 = 2(g_1 + g_2 + g_3) - 4g_3 + 2 = 2(g_1 + g_2 - g_3) + 2 = 2(g_1 + g_2 - g_2 - g_1 + n - 1) + 2 = 2n$$

By Lewittes' theorem if  $g_3 \neq g_1 + g_2$  and  $g_3 \neq g_1 + g_2 - 1$  then the ramification points of  $f_3$  are Weierstrass points. Let  $i \in \{1, 2\}$ . Then we have

$$r_i = 2g - 2 - 4g_i + 4 = 2(g_1 + g_2 + g_3) - 4g_i + 2 > 4.$$

By Lewittes' theorem any ramification point of  $f_i$  is a Weierstrass point. Hence we get our desired result. Q.E.D.

### References

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