Note on Davenport's Paper "On the series L (1)"

Teluhiko HLANO

1. Theorems

Let χ be a primitive character mod q (>1). For positive number s and non-negative integer j, we put

$$S_j(s)=\sum_{n=jq+1}^{(j+1)q}\frac{\chi(n)}{n^s}.$$

Davenport[1] shows that

THEOREM. *Let χ be a real primitive character* mod $q(>1)$.
i) *If $\chi(-1)=1$, then, for every non-negative integer j,*

$$S_j(1)>0.$$

ii) *For every integer $r>0$, there exists a character χ such that $\chi(-1)=-1$ and*

$$S_j(1)<0.$$

for every $j=1, 2, \cdots, r$.

In this note, we shall show that Davenport's method is also available for $S_j(s)$, where $0<s<1$ and integer $j>0$;

THEOREM 1. *Let χ be a real primitive character* mod *q and $\chi(-1)=1$. Then, for every integer $j>0$ and real number s, $0<s\leqq 1$, we have*

$$S_j(s)>S_{j+1}(s)>0.$$

THEOREM 2. *For every integer $r>0$ and real number s_0, where $0<s_0<1$, there exists a real primitive character χ* mod *q where $\chi(-1)=-1$,*

$$S_j(s)<0,$$

for $j=1, 2, \cdots, r$ and s, where $s_0<s\leqq 1$.

The proofs of our theorems are similar to that of Davenport.

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2. Lemmas

In this section, we shall show some lemmas used in the proofs of the theorems.

LEMMA 1. *Let a_1, a_2, a_3, and a_4 be positive real number and satisfy the conditions*

$$a_1 \leqq a_2 \leqq a_3 \leqq a_4$$

and

$$a_1 + a_4 \leqq a_2 + a_3.$$

Then, for $x>0$ and $s>0$, we have

$$\frac{1}{(x+a_1)^s} - \frac{1}{(x+a_2)^s} - \frac{1}{(x+a_3)^s} + \frac{1}{(x+a_4)^s} > 0$$

PROOF. We have

$$\begin{aligned} (2) \qquad & \frac{1}{(x+a_1)^s} - \frac{1}{(x+a_2)^s} - \frac{1}{(x+a_3)^s} + \frac{1}{(x+a_4)^s} \\ &= s\left[\int_{a_1}^{a_2} \frac{du}{(x+u)^{s+1}} - \int_{a_3}^{a_4} \frac{du}{(x+u)^{s+1}}\right] \\ &\geqq s\int_{a_1}^{a_1+a_4-a_3} \left[\frac{1}{(x+u)^{s+1}} - \frac{1}{(x+a_3-a_1+u)^{s+1}}\right] du \\ &= s(s+1)\int_{a_1}^{a_1+a_4-a_3} du \int_0^{a_3-a_1} \frac{du}{(x+u+v)^{s+2}} \\ &\geqq s(s+1)(a_4-a_3)(a_3-a_1)\frac{1}{(x+a_4)^{s+2}} \end{aligned}$$

LEMMA 2. *Let*

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} [a_m \cos 2\pi mx + b_m \sin 2\pi mx]$$

be Fourier expansion of the function

$$f(x) = \frac{1}{(j+x)^s}$$

Then we have

$$a_m = \frac{2s}{(2\pi m)^2}\left(\frac{1}{j^{s+1}} - \frac{1}{(j+1)^{s+1}}\right) - \frac{2s(s+1)(s+2)}{(2\pi m)^4}\left(\frac{1}{j^{s+3}} - \frac{1}{(j+1)^{s+3}}\right)\theta_m$$

and

$$b_m = \frac{2}{2\pi m}\left(\frac{1}{j^s} - \frac{1}{(j+1)^s}\right) - \frac{2s(s+1)}{(2\pi m)^3}\left(\frac{1}{j^{s+2}} - \frac{1}{(j+1)^{s+2}}\right)\varphi_m,$$

where θ_m and φ_m are some constants between 0 *and* 1.

PROOF. We put

$$a_m(s)=2\int_0^1 \frac{\cos 2\pi mx}{(j+x)^s}\,dx,$$

$$b_m(s)=2\int_0^1 \frac{\sin 2\pi mx}{(j+x)^s}\,dx.$$

Using the partial integration, we get

$$(3)\qquad a_m(s)=\frac{2}{2\pi m}\,\frac{\sin 2\pi mx}{(j+x)^s}\Big|_0^1+\frac{s}{2\pi m}\,b_m(s+1)$$

$$=\frac{s}{2\pi m}\,b_m(s+1),$$

$$(4)\qquad b_m(s)=-\frac{2}{2\pi m}\,\frac{\cos 2\pi mx}{(j+x)^s}\Big|_0^1-\frac{s}{2\pi m}\,a_m(s+1)$$

$$=\frac{2}{2\pi m}\left(\frac{1}{j^s}-\frac{1}{(j+1)^s}\right)-\frac{s}{2\pi m}\,a_m(s+1).$$

From (3) and (4), we have

$$(5)\qquad a_m(s)=\frac{2s}{(2\pi m)^2}\left(\frac{1}{j^{s+1}}-\frac{1}{(j+1)^{s+1}}\right)-\frac{2s(s+1)(s+2)}{(2\pi m)^3}\,b_m(s+3)$$

and

$$(6)\qquad b_m(s)=\frac{2s}{2\pi m}\left(\frac{1}{j^s}-\frac{1}{(j+1)^s}\right)-\frac{2s(s+1)}{(2\pi m)^2}\,b_m(s+2).$$

On the other hand, we get

$$b_m(s)=2\sum_{k=0}^{m-1}\int_{k/m}^{(k+1)/m}\frac{\sin 2\pi mx}{(j+x)^s}\,dx$$

$$=2\sum_{k=0}^{m-1}\int_{k/m}^{(2k+1)/2m}\sin 2\pi mx\left(\frac{1}{(j+x)^s}-\frac{1}{(j+x+m/2)^s}\right)dx\geqq 0.$$

From this formula and (6), we have

$$a_m(s),\ b_m(s)\geqq 0.$$

Hence

$$(7)\qquad 0\leqq b_m(s)\leqq\frac{2}{2\pi m}\left(\frac{1}{j^s}-\frac{1}{(j+1)^s}\right).$$

Substituting this formula from (5) to (7), we have proved the lemma.

The following lemma is well-known.

LEMMA 3. *Let χ be a real non-principal primitive character* mod q. *Then we have*

$$\sum_{n=1}^{q-1}\chi(n)e^{2\pi imn/q}=\chi(m)\sqrt{\chi(-1)q}.$$

3. Proofs of the theorems

From (1) and Lemma 2, we have

$$\begin{aligned} S_j(s) &= \sum_{n=1}^{q-1} \frac{\chi(n)}{(qj+n)^s} \\ &= \frac{1}{q^s} \sum_{n=1}^{q-1} \chi(n) f\left(\frac{n}{q}\right) \\ &= \sum_{m=1}^{\infty} \left[a_m \left(\sum_{n=1}^{q-1} \chi(n) \cos 2\pi mn/q \right) + b_m \left(\sum_{n=1}^{q-1} \chi(n) \sin 2\pi mn/q \right) \right]. \end{aligned}$$

By Lemma 3, we know that the coefficients of b_m is zero when $\chi(-1)=1$ and that those of a_m is zero when $\chi(-1)=-1$. Hence

$$(8) \qquad S_j(s) = \begin{cases} \dfrac{1}{q^{s-1/2}} \displaystyle\sum_{m=1}^{\infty} a_m(s)\chi(m) \text{ if } \chi(-1)=1 \\ \dfrac{1}{q^{s-1/2}} \displaystyle\sum_{m=1}^{\infty} b_m(s)\chi(m) \text{ if } \chi(-1)=-1 \end{cases}$$

Now we prove Theorem 1.
By Lemma 2, we get

$$(9) \quad S_j(s) q^{s-1/2} = \frac{s}{2\pi^2} L(2, \chi) \left(\frac{1}{j^{s+1}} - \frac{1}{(j+1)^{s+1}} \right) - \frac{s(s+1)(s+2)}{8\pi^4} \left(\frac{1}{j^{s+3}} - \frac{1}{(j+1)^{s+3}} \right) J_1,$$

where

$$(10) \qquad J_1 = \sum_{m=1}^{\infty} \theta_m \chi(m) \frac{1}{m^4}.$$

From the definition of θ_m, we have

$$(11) \qquad |J_1| \leqq \zeta(4).$$

Using (9)-(11), we get

$$S_j(s) q^{s-1/2} \geqq \frac{s}{2\pi^2} L(2, \chi) \left(\frac{1}{j^{s+1}} - \frac{1}{(j+1)^{s+1}} \right) - \frac{s(s+1)(s+2)}{8\pi^4} \left(\frac{1}{j^{s+3}} - \frac{1}{(j+1)^{s+3}} \right) \zeta(4)$$

and

$$\begin{aligned} & S_{j+1}(s) q^{s-1/2} \\ & \leqq \frac{s}{2\pi^2} L(2, \chi) \left(\frac{1}{(j+1)^{s+1}} - \frac{1}{(j+2)^{s+1}} \right) + \frac{s(s+1)(s+2)}{8\pi^4} \left(\frac{1}{(j+1)^{s+3}} - \frac{1}{(j+2)^{s+3}} \right) \zeta(4). \end{aligned}$$

Since

$$(12) \qquad L(2, \chi) \geqq \prod_p \left(1 + \frac{1}{p^2} \right)^{-1} = \zeta(4) \zeta(2)^{-1},$$

we have

$$(13)\qquad \begin{aligned}&(S_j(s)-S_{j+1}(s))q^{s-1/2}\\ &\geqq \frac{s}{2\pi^2}L(2,\chi)\left(\frac{1}{j^{s+1}}-\frac{2}{(j+1)^{s+1}}+\frac{1}{(j+2)^{s+1}}\right)\\ &\quad-\frac{s(s+1)(s+2)}{2\pi^4}\left(\frac{1}{j^{s+3}}-\frac{1}{(j+2)^{s+3}}\right)\zeta(4)\\ &\geqq \frac{s}{8\pi^4}\zeta(4)\Big[24\left(\frac{1}{j^{s+1}}-\frac{2}{(j+1)^{s+1}}+\frac{1}{(j+2)^{s+1}}\right)\\ &\quad-(s+1)(s+2)\left(\frac{1}{j^{s+3}}-\frac{1}{(j+2)^{s+3}}\right)\Big]\end{aligned}$$

Using (2), the right hand side of (13) can be estimated by

$$\frac{\zeta(4)}{8\pi^4}\,s(s+1)(s+2)\left[\frac{25}{(j+2)^{s+3}}-\frac{1}{j^{s+3}}\right]$$

from below. Since

$$\left(1+\frac{2}{j}\right)^{s+3}\leqq 2^{s+3}\leqq 2^4<25,$$

for $j\geqq 2$ and $0<s\leqq 1$, we have proved the theorem for this case. For $j=1$, the right hand side of (13) will be estimated as follows: for $s\geqq 1/2$

$$24\left(1-\frac{2}{2^{s+1}}+\frac{1}{3^{s+1}}\right)-(s+1)(s+2)\left(1-\frac{1}{3^{s+3}}\right)>24-\frac{24}{\sqrt{2}}-6=6(3-\sqrt{8})>0,$$

while, for $s\leqq 1/2$

$$\frac{24}{3^{3/2}}-(s+1)(s+2)\geqq\frac{32-15\sqrt{3}}{4\sqrt{3}}>0.$$

Now Theorem 1 has been proved.

Next we prove Theorem 2. By Lemma 2, we have

$$(14)\qquad S_j(s)q^{s-1/2}=\frac{1}{\pi}L(1,\chi)\left(\frac{1}{j^s}-\frac{1}{(j+1)^s}\right)-\frac{s(s+1)}{4\pi^3}\left(\frac{1}{s+2}-\frac{1}{(j+1)^{s+2}}\right)J_{-1},$$

where

$$(15)\qquad J_{-1}=\sum_{m=1}^{\infty}\phi_m\chi(m)\frac{1}{m^3}$$

From Lemma 2, we have

$$(17)\qquad |J_{-1}-\phi_1|\leqq\zeta(3)-1$$

and

$$(18)\qquad \phi_1=b_1(s+2)/\frac{1}{\pi}\left(\frac{1}{j^{s+2}}-\frac{1}{(j+1)^{s+2}}\right).$$

Now we need

LEMMA 6. *For $k \geqq 1$, we put*

$$G(k)=b_1(k)/\frac{1}{\pi}\left(\frac{1}{j^k}-\frac{1}{(j+1)^k}\right).$$

Then we have

$$1 \geqq G(k) \geqq \frac{8}{5}\left(\frac{4}{7}\right)^k.$$

PROOF. We have

$$\int_0^1 \frac{\cos 2\pi x}{(j+x)^{k+1}}\,dx$$
$$=\int_0^{1/4} \cos 2\pi x\left(\frac{1}{(j+x)^{k+1}}-\frac{1}{(j+1/2-x)^{k+1}}-\frac{1}{(j+1/2+x)^{k+1}}+\frac{1}{(j+1-x)^{k+1}}\right)dx.$$

It follows from Lemma 1 that the above integrand is non-negative. Thus

$$(19)\qquad \int_0^1 \frac{\cos 2\pi x}{(j+x)^{k+1}}\,dx$$
$$\leqq \int_0^{1/4}\left(\frac{1}{(j+x)^{k+1}}-\frac{1}{(j+1/2-x)^{k+1}}-\frac{1}{(j+1/2+x)^{k+1}}+\frac{1}{(j+1-x)^{k+1}}\right)dx$$
$$=\frac{1}{k}\left(\frac{1}{j^k}-\frac{2}{(j+1/4)^k}+\frac{2}{(j+3/4)^k}-\frac{1}{(j+1)^k}\right).$$

Substituting (19) to the definition of $G(k)$ and (4), we have

$$(20)\qquad 1 \geqq G(k) \geqq 2\left(\frac{1}{(j+1/4)^k}-\frac{1}{(j+3/4)^k}\right)\Big/\left(\frac{1}{j^k}-\frac{1}{(j+1)^k}\right).$$

Using the mean value theorem for the function x^k, we have

$$\frac{1}{(j+1/4)^k}-\frac{1}{(j+3/4)^k} \geqq k\left(\frac{1}{(j+1/4)}-\frac{1}{(j+3/4)}\right)\left(\frac{1}{j+3/4}\right)^{k-1}$$

and

$$\frac{1}{j^k}-\frac{1}{(j+1)^k} \leqq k\left(\frac{1}{j}-\frac{1}{j+1}\right)\left(\frac{1}{j}\right)^{k-1}.$$

Hence (20) implies

$$1 \geqq G(k) \geqq \frac{j(j+1)}{(j+1/4)(j+3/4)}\left(\frac{j}{j+3/4}\right)^{k-1} \geqq \frac{8}{5}\left(\frac{4}{7}\right)^k.$$

Lemma 6 implies

$$1 \geqq \varphi_1 \geqq \frac{8}{5}\left(\frac{4}{7}\right)^{s+2}.$$

From (15), (17) and Lemma 2, we have

$$S_j(s)q^{s-1/2} \leqq \frac{1}{\pi} L(1,\ \chi)\left(\frac{1}{j^s}-\frac{1}{(j+1)^s}\right)-\frac{s(s+1)}{4\pi^3}\left(\frac{1}{j^{s+2}}-\frac{1}{(j+1)^{s+2}}\right)(\varphi_1-\zeta(3)+1).$$

On the other hand, we get

$$\begin{aligned}\varphi_1-\zeta(3)+1 &\geqq \frac{8}{5}\left(\frac{4}{7}\right)^{s+2}-(\zeta(3)-1)\\ &\geqq \frac{8}{5}\left(\frac{4}{7}\right)^{3}-\left(\frac{1}{8}+\int_2^\infty \frac{dt}{t^3}\right)\\ &\geqq \frac{512}{1715}-\frac{1}{4}=c>0.\end{aligned}$$

Hence there exists some positive constant c such that

$$S_j(s)q^{s-1/2} \leqq \frac{1}{\pi} L(1,\ \chi)\left(\frac{1}{j^s}-\frac{1}{(j+1)^s}\right)-\frac{s(s+1)}{4\pi^3}c\left(\frac{1}{j^{s+2}}-\frac{1}{(j+1)^{s+2}}\right). \tag{23}$$

As Davenport[1], we know that there exists a character such that $L(1,\ \chi)$ is small as possible. This implies the theorem.

References

[1] Davenport, H., On the series of $L(2)$, *J. London Math. Soc.* **24** (1949) 229-233
[2] Davenport, H., *Multiplicative Number Theory, 2nd Ed.* Springer Verlag 1980