On Non-Weierstrass Gap Sequences

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Abstract

Let G be a finite subset of the additive semigroup N of non-negative integers whose complement in N becomes a subsemigroup of N. Then we say that the set G is a *gap sequence of genus* g where g is the number of G. In this paper we research on the gap sequences G of genus g such that the number of the set $\{r+s \mid r, s \in G\}$ is larger than $3g$-3. It is well-known that the above gap sequence is not that of a point of any curve, where a *curve* means a complete non-singular irreducible 1-dimensional algebraic variety over an algebraically closed field k of characteristic zero.

Introduction

Let C be a curve. For any point P of C, a non-negative integer n is called a *gap at* P if

$$H^0(C, O_C((n-1)P)) = H^0(C, O_C(nP))$$

We denote by $G(P)$ the set of gaps at P. A gap sequence G is said to be *Weierstrass* if there exists a point P of a curve satisfying $G=G(P)$. The following was a long-standing problem: *are all gap sequences Weierstrass*? But Buchweitz finally showed that not every gap sequence is Weierstrass[1]. At present the following is an open problem: *numerically describe a necessary and sufficient condition on a gap sequence G such that G is Weierstrass.* Buchweitz gave the following necessary condition: *if G is Weierstrass, then the number of the set* $\{r+s \mid r, s \in G\}$ *is less than or equal to* $3g$-3 *where g is the number of G.* In this paper using the above we give many examples of non-Weierestrass gap sequences. In the last section we give the table of the number of the gap sequences of genus g and that of the number of the gap sequences of genus g such that the number of the set $\{r+s \mid r, s \in G\}$ is larger than $3g$-3. This table is the output of a TURBO C program which was run on the author's PC-9801. Finally the author would like to thank Dr. S. Tsuyumine and Mr. A. Seyama, because the former wrote a BASIC program which motivated the author to write this paper and the latter wrote the above TURBO C program.

§1. Examples of Non-Weierstrass Gap Sequences.

Definition. Let $G=\{a_1, a_2, \cdots a_g\}$ be a gap sequence of genus g with $a_i < a_j$ if $i<j$. For any i, we set $\alpha_i = a_i - i$. Then we say that $(\alpha_1, \alpha_2, \cdots, \alpha_g)$ is the *Schubert index associated to* G, which is denoted by $S(G)$.

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Definition. Let G be a gap sequence of genus g. If the number of the set $\{r+s \mid r, s \in G\}$ is less than or equal to $3g$-3, then G is said to be *semi-Weierstrass*. It is known that any Weierstrass gap sequence is semi-Weierstrass.

The example of Buchweitz is the case $n=12$, $m=3$ of the following:

Example 1. For any two integers n and m with $n \geqq 4m$ and $m \geqq 3$, let G be the gap sequence of genus $n+m+1$ with

$$S(G)=(0, \cdots, 0, n-2m, n-2m+1, \cdots, n-2m+m-2, n-m, n-m).$$

Then G is not semi-Weierstrass[2].

Example 2. For any two integers n and m with $n \geqq 4m$ and $m \geqq 3$, let G be th gap sequence of genus $n+m+1$ with

$$S(G)=\{0, \cdots, 0, n-2m, n-2m, n-2m+2, n-2m+2+1, \cdots, n-2m+2+m-2).$$

Then G is not semi-Weierstrass.

Proof. We have

$G=\{1, 2, \cdots, n, 2n-2m+1, 2n-2m+2, 2n-\{2(m-3)+1\}, 2n-\{2(m-4)+1\}, \cdots, 2n-1, 2n+1\}$.

We set $A=\{(r-1)+(s-1) \mid r, s \in G\}$. It is easily seen that the set A contains $0, 1, \cdots, 3n-1$. The elements of the rest of A are of type $i+j$ where i and j belong to the set

$$\{2n-2m, 2n-(2m-1)\} \cup \{2n-2l \mid 0 \leqq l \leqq m-2\}.$$

Hence we obtain

$$^{\#}A=3n+3m+1=3(n+m+1)-2>3(n+m+1)-3$$

where $^{\#}A$ denotes the number of the set A. *Q.E.D.*

Example 3. For any two integers n and m with $n \geqq 4m$ and $m \geqq 3$, let G be the gap sequence of genus $n+m+2$ with

$$S(G)=(0, \cdots, 0, n-2m, n-2m, n-2m+1, \cdots, n-2m+m-1, n-2m+m-1).$$

Then G is not semi-Weierstrass. In this case the number of the set $\{r+s \mid r, s \in G\}$ is equal to $3(n+m+2)-3+m-2$.

Proof. We have

$$G=\{1, 2, \cdots, n, 2n-2m+1, 2n-2(m-1), 2n-2(m-2), \cdots, 2n, 2n+1\}.$$

It is easily seen that the set $A=\{(r-1)+(s-1) \mid r, s \in G\}$ contains $0, 1, \cdots, 3n-1$. The elements of the rest of A are of type $i+j$ where i and j belong to the set

$$\{2n-2m,\ 2n\}\cup\{2n-(2l+1)\mid 0\leqq l\leqq m-1\}.$$

Hence we obtain

$$^{\#}A=3n+4m+1=3(n+m+2)-3+m-2. \qquad Q.E.D.$$

Example 4. For any two integers n and m with $n\geqq 4m$ and $m\geqq 3$, let G be the gap sequence of genus $n+m+2$ with

$$S(G)=(0,\ \cdots,\ 0,\ 1,\ n-2m+1,\ n-2m+1,\ n-2m+3,\ n-2m+3+1,\ \cdots,\ n-2m+3+m-2).$$

Then G is not semi-Weierstrass.

Proof. We have

$$G=\{1,\ 2,\ \cdots,\ n,\ n+2,\ 2n-2m+3,\ 2n-2m+4,\ 2n-\{2(m-4)+1\},\ 2n-\{2(m-5)+1\},\ \cdots,\ 2n+1,\ 2n+3\}.$$

The set $A=\{(r-1)+(s-1)\mid r,\ s\in G\}$ contains $0,\ 1,\ \cdots,\ 3n+1,\ 3n+3$. The elements of the rest of A are of type $i+j$ where i and j belong to the set

$$\{2n-2m+2,\ 2n-2m+3\}\cup\{2n-2l\mid -1\leqq l\leqq m-3\}.$$

Hence we obtain

$$^{\#}A=3n+3+3m+1=3(n+m+2)-2. \qquad Q.E.D.$$

Example 5. For any two integers n and m with $n\geqq 4m$ and $m\geqq 3$, let G be the gap sequence of genus $n+m+2$ with

$$S(G)=\{0,\ \cdots,\ 0,\ 1,\ n-2m,\ n-2m+3,\ n-2m+3,\ n-2m+3+1,\ \cdots,\ n-2m+3+m-2\}.$$

If $n\neq 4m+1$, then G is not semi-Weierstrass.

Proof. We have

$$G=\{1,\ 2,\ \cdots,\ n,\ n+2,\ 2n-2m+2,\ 2n-2m+6,\ 2n-\{2(m-4)+1\},\ 2n-\{2(m-5)+1\},\ \cdots,\ 2n+1,\ 2n+3\}.$$

The set $A=\{(r-1)+(s-1)\mid r,\ s\in G\}$ contains $0,\ 1,\ \cdots,\ 3n+1$. If $n=4m$, then A contains $3n+2$ and $3n+3$. If $n>4m$, then $3n+2\in H$ and $3n+3\in G$, where H is the complement of G in $\boldsymbol{N}$. We see easily that in the case $n\neq 4m+1$ the number of A is equal to $3(n+m+2)-2$.

Q.E.D.

Similarly we obtain the following:

Example 6. For any two integers n and m with $n\geqq 4m$ and $m\geqq 3$, let G be the gap sequence of genus $n+m+2$ with

$$S(G)=(0,\ \cdots,\ 0,\ 1,\ n-2m,\ n-2m+1,\ \cdots,\ n-2m+m-2,\ n-2m+m-2,\ n-2m+m+1).$$

If $n \neq 4m+1$, then G is not semi-Weierstrass.

The non-semi-Weierstrass gap sequences of genus 16 or 17 are special cases of the above six Examples.

§2. Examples of Semi-Weierstrass Gap Sequences.

In this section we show that a gap sequence with a "simple" Schubert index is semi-Weierstrass.

Proposition. *For any integer $n \geqq 3$, let G be a gap sequence of genus $n+3$ with $S(G) = (0, \cdots, 0, a_{n+1}, a_{n+2}, a_{n+3})$. Then S is semi-Weierstrass.*

Proof. We have

$$G = \{1, 2, \cdots, n, n+1+a_{n+1}, n+2+a_{n+2}, n+3+a_{n+3}\}.$$

It is known that $n+3+a_{n+3} \leqq 2g-1$ where g is the number of G. Hence we get $a_{n+3} \leqq n+2$.

Let $a_{n+3} \leqq n-2$. In view of $n-1+n+2+a_{n+3} \leqq 3n-1$, we see that $^{\#}A \leqq 3g-3$, where $^{\#}A$ is the number of the set

$$A = \{(r-1)+(s-1) \mid r, s \in G\}.$$

Hence G is semi-Weierstrass. We may assume that $n-1 \leqq a_{n+3} \leqq n+2$.

Let $a_{n+1} = a_{n+2} = 0$. The we have

$$G = \{1, 2, \cdots, n, n+1, n+2, n+3+a_{n+3}\}.$$

It is easily seen that $^{\#}A \leqq 3g-3$.

Let $a_{n+1} = 0$ and $a_{n+2} > 0$. If $a_{n+3} \leqq n$, we see that G is semi-Weierstrass. If $a_{n+3} = n+1$, we have

$$G = \{1, 2, \cdots, n, n+1, n+2+a_{n+2}, 2n+4\}.$$

Let H be the complement of G in $\boldsymbol{N}$. Then $n+2 \in H$ and $2n+4 \in G$, which is a contradiction. If $a_{n+3} = n+2$, we have

$$G = \{1, 2, \cdots, n, n+1, n+2+a_{n+2}, 2n+5\}.$$

Hence $n+2 \in H$, which implies that $n+3 \in G$. Therefore $a_{n+2} = 1$. Hence G is semi-Weierstrass.

Let $a_{n+1} > 0$. Then we have $n+1 \in H$. If $a_{n+3} = n-1$, then $2n+2 = n+3+a_{n+3} \in G$, which is a contradiction. If $a_{n+3} = n$, then we have

$$(n+1)+(n+1+a_{n+2}) \leqq 3n+1.$$

Hence G is semi-Weierstrass. If $a_{n+3} = n+1$, then $n+2$ and $n+3$ belong to G. Hence G is semi-Weierstrass. If $a_{n+3} = n+2$, then $n+4 \in G$. Moreover, we have $n+2 \in G$ or $n+3 \in G$. If $n+2 \in G$, then we have $\{r-1 \mid r \in G\} = \{0, \cdots, n-1, n+3, 2n+4\}$. In view of $n \geqq$

3, we have $2(n+3)\leqq 3n+3$. Moreover, $2n+3$ does not belong to A, which implies that ${}^{\#}A \leqq 3n+3+3=3g-3$. If $n+3\in G$, then by the similar way we have ${}^{\#}A\leqq 3g-3$. *Q.E.D.*

§3. On Gap Sequences of Genus $n+4$ with Indices $(0, \cdots, 0, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4})$.

In this section let G be a gap sequence of genus $g=n+4$ with a Schubert index $S(G)=(0, \cdots, 0, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4})$, and let H be the complement of G in $\boldsymbol{N}$. Then

$G=\{1, 2, \cdots, n, n+1+\alpha_{n+1}, n+2+\alpha_{n+2}, n+3+\alpha_{n+3}, n+4+\alpha_{n+4}\}$ and
$G_{-1}=\{r-1 \mid r\in G\}=\{0, 1, \cdots, n-1, n+\alpha_{n+1}, n+1+\alpha_{n+2}, n+2+\alpha_{n+3}, n+3+\alpha_{n+4}\}$.

We set $A=\{(r-1)+(s-1) \mid r, s\in G\}$. We assume that $n\geqq 12$ and $\alpha_{n+1}\geqq 1$.

Lemma 1. *If G is not semi-Weierstrass, then $\alpha_{n+4}\geqq n-3$.*

Proof. Assume that $\alpha_{n+4}\leqq n-4$. Since we have $n-1+n+3+\alpha_{n+4}\leqq 3n-2$, we see that A is contained in the set

$$[0, 3n-2]\cup\{2n+a+b \mid a, b\in\{\alpha_{n+1}, 1+\alpha_{n+2}, 2+\alpha_{n+3}, 3+\alpha_{n+4}\}\}.$$

Hence we have

$${}^{\#}\mathrm{A}\leqq 3n-1+10=3n+9=3g-3,$$

which is a contradiction. *Q.E.D.*

Lemma 2. *If G is not semi-Weierstrass, then $\alpha_{n+1}=n-m$ with $m\geqq 6$.*

Proof. Set $\alpha_{n+1}=n-m$ with an integer m. Assume that $m\leqq 5$. Then $n+1+\alpha_{n+1}\geqq n+1+n-5$. Hence $\{n+1, \cdots, n+1+n-6\}\subset H$. In view of $n\geqq 12$, we get

$$n+1+n-6+n+1>2g-1\geqq n+4+\alpha_{n+4},$$

because any element r of G is less than $2g$. Therefore we get $n+4+\alpha_{n+4}<2(n+1)$, which implies that $n+4+\alpha_{n+4}<2(n+1)$, which implies that $\alpha_{n+4}\leqq n-3$. By Lemma 1, $\alpha_{n+4}=n-3$. Since $n-m<1+\alpha_{n+2}<2+\alpha_{n+3}<n$, we have $n-m\leqq n-3$, wich implies that $3\leqq m$. Therefore we have

$${}^{\#}A=3n+{}^{\#}\{a+b \mid a, b\in\{n-\alpha_{n+1}, n-1-\alpha_{n+2}, n-2-\alpha_{n+3}, 0\}\}\leqq 3n+9=3g-3,$$

which is a contradiction. *Q.E.D.*

Lemma 3. *Let G be non-semi-Weierstrass. Then $\alpha_{n+1}=n-m$ with $m\leqq[n/2]$ if and only if $\alpha_{n+4}=n-3$.*

Proof. Assume that $\alpha_{n+4}=n-3$ and that $\alpha_{n+1}=n-m$ with $m\geqq[n/2]+1$. Then we have

$$A=\{0, 1, \cdots, 3n-1\}\cup\{2n+a+b \mid a, b\in \{a_{n+1}, 1+a_{n+2}, 2+a_{n+3}, 3+a_{n+4}\}\}.$$

In view of $m\geqq[n/2]+1$, we have $^{\#}A\leqq 3n+9=3g-3$, which is a contradiction. Assume that $a_{n+1}=n-m$ with $m\leqq[n/2]$ and that $a_{n+4}\neq n-3$. Hence by Lemma 1 we have $a_{n+4}\geqq n-2$. Since we have $n+4+a_{n+4}=2n+2+r$ with $1\leqq r\leqq 5$, we obtain $n+1+a_{n+4}\leqq n+1+r$ because of $n+1\in H$. Hence we get $n-m=a_{n+1}\leqq r\leqq 5$, which implies that $n-5\leqq m\leqq[n/2]$. This contradicts $n\geqq 12$. *Q.E.D.*

Lemma 4. *We have* $a_{n+4}=n-3$.

Proof. Assume that $a_{n+4}\neq n-3$. Then we have $n-2\leqq a_{n+4}\leqq n+3$. Let $a_{n+4}=n-2$. Then $n+4+a_{n+4}=2(n+1)\in H$, which is a contradiction.

Let $a_{n+4}=n-1$. Then we must have $a_{n+1}=1$. Hance we get

$$G_{-1}=\{0, 1, \cdots, n-1, n+1, n+1+a_{n+2}, n+2+a_{n+3}, 2n+2\}.$$

In view of $a_{n+2}\leqq a_{n+4}$, we have $n+1+n+1+a_{n+2}\leqq 3n+1$. Moreover, we have $n+1+n+2+a_{n+3}\leqq 3n+1$. Hance we have $^{\#}A\leqq 3g-3$, which is a contradiction.

Let $a_{n+4}=n$. Then we get

$$G_{-1}=\{0, 1, \cdots, n-1, n+1, n+2, n+2+a_{n+3}, 2n+3\}$$

Assume that $3n+2<n+1+n+2+a_{n+3}$. Then we have $a_{n+3}=n$. Hence we get $^{\#}A\leqq 3n+3+6=3g-3$, which is a contradiction.

Let $a_{n+4}=n+1$. Then we have $n+1+1\in G$ or $n+1+2\in G$. If $n+1+1\in G$ and $n+1+2\in G$, then we obtain $a_{n+1}=a_{n+2}=a_{n+3}=1$. Then $^{\#}A\leqq 3g-3$. If $n+1+1\in G$ and $n+1+2\in H$, then $a_{n+1}=1$ and $a_{n+2}=2$, which implies that $^{\#}A\leqq 3g-3$. If $n+1+1\in H$ and n $+1+2\in G$, then $a_{n+1}=2$ and $a_{n+4}=2$. Then we have $n+2+(n+2+a_{n+3})\leqq 3n+3$, which implies that $^{\#}A\leqq 3g-3$.

Let $a_{n+4}=n+2$. Then $n+1+1\in G$ or $n+1+3\in G$. If $n+1+1\in G$, then $a_{n+1}=a_{n+2}=1$ and $a_{n+3}=2$, which implies that $^{\#}A\leqq 3g-3$. If $n+1+3\in G$, then $a_{n+1}=a_{n+2}=a_{n+3}=2$, which implies that $^{\#}A\leqq 3g-3$.

Let $a_{n+4}=n+3$. Then $n+1+5\in G$. If $n+1+1\in G$, then $n+1+4\in H$. Then we have

$$G_{-1}=\{0, 1, \cdots, n-1, n+1, a, n+5, 2n+6\}$$

where a is $n+2$ or $n+3$, which implies that $^{\#}A\leqq 3g-3$. For the rest, by the similar way to the above we have $^{\#}A\leqq 3g-3$. *Q.E.D.*

Proposition. *Let n and a_{n+1} be integers with $n\geqq 12$ and $a_{n+1}\geqq 1$. Let G be a gap sequence of genus $g=n+4$ with a Schubert index $(0, \cdots, a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4})$. Then the following are equivalent :*

(1) *G is not semi-Weiserstrass,*

(2) $a_{n+1}=n-m$ *with* $6\leqq m\leqq[n/2]$, $a_{n+4}=n-3$ *and* ${}^{\#}\{a+b \mid a, b\in\{0, 1+a_{n+2}-a_{n+1}, 2+a_{n+3}-a_{n+1}, n-a_{n+1}\}\}={}^{\#}\{c+d \mid c, d\in\{a_{n+1}, 1+a_{n+2}, 2+a_{n+3}, n\}\}=10$.
In this case, we have ${}^{\#}A=3g-2$.

Proof. First we show that (1) implies (2). By Lemmas 2, 3 and 4, we have $a_{n+1}=n-m$ with $6\leqq m\leqq[n/2]$ and $a_{n+4}=n-3$. Then we get

$$G_{-1}=\{0, 1, \cdots, n-1, 2n-m, n+1+a_{n+2}, n+2+a_{n+3}, 2n\}.$$

In view of $m\leqq[n/2]$, we obtain $2(2n-m)>n-1+2n$. Hence we have

$$A=\{0, 1, \cdots, 3n-1\}\cup\{p+q \mid p, q\in\{2n-m, n+1+a_{n+2}, n+2+a_{n+3}, 2n\}\},$$

which implies (2).

Assume that

$$G_{-1}=\{0, \cdots, n-1, 2n-m, n+1+a_{n+2}, n+2+a_{n+3}, 2n\}$$

with $6\leqq m\leqq[n/2]$. Then we have

$$A=\{0, 1, \cdots, 3n-1\}\cup\{p+q \mid p, q\in\{2n-m, n+1+a_{n+2}, n+2+a_{n+3}, 2n\}\},$$

which implies that ${}^{\#}A=3g-2$. *Q.E.D.*

§4. The Non-semi-Weierstrass Gap Sequences of Genus 16 or 17 and the Table of the Number of Non-semi-Weierstrass Gap Sequences.

By the outputs of a BASIC program which was written by Dr. S. Tsuyumine, we know that any gap sequences of genus $g\leqq 15$ is semi-Weierstrass. Moreover, the outputs of the BASIC program show the following.

Example. (1) The Schubert indices of the non-semi-Weierstrass gap sequences of genus 16 are $(0, \cdots, 0, 6, 7, 9, 9)$ and $(0, \cdots, 0, 6, 6, 8, 9)$.

(2) The Schubert indices of the non-semi-Weierstrass gap sequences of genus 17 are $(0, \cdots, 0, 6, 6, 7, 8, 8)$, $(0, \cdots, 0, 1, 7, 7, 9. 10)$, $(0, \cdots, 0, 1, 6, 9, 9, 10)$, $(0, \cdots, 0, 1, 6, 7, 7, 10)$, $(0, \cdots, 0, 0, 7, 8, 10, 10)$ and $(0, \cdots, 0, 0, 7, 7, 9, 10)$.

Let N_g be the number of the gap sequences of genus g and L_g the number of the non-semi-Weierstrass gap sequences of genus g. The following table is the outputs of a TURBO C program was written by Mr. A. Seyama.

g	N_g	L_g	$100L_g/N_g$
16	4806	2	0.0416
17	8045	6	0.0746
18	13467	15	0.1114
19	22464	31	0.1380
20	37396	67	0.1792
21	62194	145	0.2331

22	103246	293	0.2838
23	170963	542	0.3170
24	282828	1053	0.3723
25	467224	1944	0.4161
26	770832	3591	0.4659
27	1270267	6584	0.5183
28	2091030	11871	0.5677
29	3437839	20987	0.6105
30	5646773	37598	0.6658
31	9266788	66330	0.7158
32	15195070	116501	0.7667
33	24896206	203300	0.8166
34	40761087	353978	0.8684
35	66687201	615762	0.9234
36	109032500	1058418	0.9707
37	178158289	1819323	1.0212

It seems to me that if the genus increases by 1, then the percentage of the non-semi-Weierstrass gap sequences does by about 0.05.

Conjecture. We have $\lim_{g\to\infty} L_g/N_g=1$.

References

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