On Primitive Schubert Indices

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Abstract

Let $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ be a Schubert index of genus g, i.e., $\alpha_0, \alpha_1, \cdots, \alpha_{g-1}$ are non-negative integers with $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{g-1} \leq g-1$. We say that α is *Weierstrass* if it is the Schubert index $\alpha(P)$ of a pointed curve (C, P) of genus g where a *curve* means a complete non-singlar 1-dimensional algebraic variety over an algebraically closed field k of characteristic 0. Moreover, it is said that α is *dimensionally proper* if there exists a pointed curve (C, P) with $\alpha(P)=\alpha$ such that the moduli space $\mathscr{C}_\alpha$ of pointed curves with Schubert index α has codimension $w(\alpha)$ in the moduli space $\mathscr{M}_{g,1}$ of pointed curves of genus g, locally at (C, P). In this paper we shall give examples of Weierstrass (resp. dimensionally proper) primitive Schubert indices of weight $\geq g$. We note that a Schubert index $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ is primitive if and only if $2(i_0+1)>g+\alpha_{g-1}$ where $\alpha_{i_0}=0$ and $\alpha_{i_0+1}\neq 0$. Moreover, examples of primitive Schubert indices which are not dimensionally proper are given.

Introduction

Let $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ be a Schubert index of genus g. Then $H(\alpha)$ denotes the complement of the set $\{\alpha_i+i+1 \mid i=0, 1, \cdots, g-1\}$ in the additive semigroup $\mathbf{N}$ of non-negative integers. It is said that α *satisfies the semigroup condition* if $H(\alpha)$ becomes a subsemigroup of $\mathbf{N}$. Recall that Schubert indices of genus g are partially ordered by $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$, $i=0, \cdots, g-1$ where $\alpha=(\alpha_0, \cdots, \alpha_{g-1})$ and $\beta=(\beta_0, \cdots, \beta_{g-1})$. We say that a Schubert index α is *primitive* if every Schubert index β with $\beta \leq \alpha$ satisfies the semigroup condition, which is equivalent to $2(i_0+1)>\alpha_{g-1}+g$ where $\alpha_{i_0}=0$ and $\alpha_{i_0+1}\neq 0$[1]. In this case, i_0+1 (resp. $\alpha_{g-1}+g$) is called the *first non-gap* (resp. the *last gap*) of α. Let C be a curve of genus g. For any point P of C, a non-negative integer n is called a *gap at P* if

$$h^0(C, \mathcal{O}_C((n-1)P))=h^0(C, \mathcal{O}_C(nP)).$$

Then the number of gaps at P is equal to g. If we set $\alpha_{i-1}(P)=m_i-1$ for $i=1, \cdots, g$ where $m_1<m_2<\cdots<m_g$ are the gaps at P, then $\alpha(P)=(\alpha_0(P), \cdots, \alpha_{g-1}(P))$ is a Schubert index of genus g satisfying the semigroup condition. But the converse is false. In fact Buchweitz first showed that not every Schbert index α satisfying the semigroup condition has a pointed curve (C, P) with $\alpha(P)=\alpha$, i.e., not every α is Weierstrass[2]. On the other hand, let $\mathscr{M}_{g,1}$ be the moduli space of pointed curves of genus g and for any Schubert index α of genus g we may define a locally closed subset of $\mathscr{M}_{g,1}$ by $\mathscr{C}_\alpha=\{(C, P)\in\mathscr{M}_{g,1} \mid \alpha(P)=\alpha\}$. If α is Weierstrass, then the weight $w(\alpha)$ of α gives an upper bound for the codimension of any component of $\mathscr{C}_\alpha$. We say that α is *dimensionally proper* if there is a component of $\mathscr{C}_\alpha$ which is of codimension $w(\alpha)$. Eisenbud-Harris (resp. the author) showed that any

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Schubert index of weight $\leq g-2$ (resp. of weight $g-1$) is dimensionally proper[1),3)].

In this paper we are devoted to primitive Schubert indices of genus g and of weight $\geq g$. First we shall give examples of dimensionally proper Schubert indices of weight g in § 1. In § 2 we shall give examples of primitive Schubert indices which are not dimensionally proper. Lastly primitive Schubert indices of genus $g \leq 8$ are investigated in § 3. The author would like to thank Dr. S. Tsuyumine for writing a TURBO C program useful for calculating examples of primitive Schubert indices with a fixed weight.

§ 1. On Primitive Schubert Indices of Genus g and of Weight g.

Using the result of Eisenbud-Harris[1)] we get the following:

Proposition 1. *Assume that for any even number $h \geq 6$ the Schubert index $\alpha(h)=(0^{h/2+1}, 2^{h/2-3}, 3^2)$ is dimensionally proper. Then any primitive Schubert index of genus g and of weight g is dimensionally proper.*

Proof. Let $\beta=(0^{g-n}, \beta_1, \cdots, \beta_n)$ be a primitive Schubert index of genus g and of weight g with $\beta_{n-1} \geq 2$. Suppose that $\beta_{n-1}=2$. Since β is primitive and of weight g, we have $2(g-n+1)>g+\beta_n$, which implies that $2(n-1)+\beta_n-2n+2-\beta_n>0$. This is a contradiction. Hence $\beta_{n-1} \geq 3$. Then we have a sequence

$$\begin{aligned}&\gamma^{(0)}=\alpha(2n+2)=(0^{n+2}, 2^{n-2}, 3) \longrightarrow \gamma^{(1)}=(0^{n+3}, 2^{n-1}, 3, 4) \longrightarrow \cdots \longrightarrow \\ &\gamma^{(\beta_n-3)}=(0^{n+\beta_{n}-1}, 2^{n-2}, 3, \beta_n) \longrightarrow \cdots \longrightarrow \gamma^{(\beta_n+\beta_{n-1}-6)} \\ &=(0^{n+\beta_n+\beta_{n-1}-4}, 2^{n-2}, \beta_{n+1}, \beta_n) \longrightarrow \cdot \cdot \longrightarrow \gamma^{(g-2n-2)}=\beta=(0^{g-n}, \beta_1, \cdots, \beta_n)\end{aligned}$$

where $w(\gamma^{(i+1)})=w(\gamma^{(i)})+1$ for $i=0, \cdots, g-2n-2$ and all Schubert indices in the above are primitive[3)]. It follows from the result of Eisenbud-Harris[1),3)] that β is dimensionally proper. Next we suppose that $\beta_1=1$. Then β_n must be larger than 1. Hence we have $\beta_1=\cdots=\beta_l=1$ and $\beta_{l+1} \geq 2$ for some $1 \leq l \leq n-1$. There is a sequence

$$\begin{aligned}&\gamma=(0^{g-n}, \beta_{l+1}, \cdots, \beta_n) \longrightarrow (0^{g-n}, 1, \beta_{l+1}, \cdots, \beta_n) \longrightarrow \cdots \longrightarrow \\ &(0^{g-n}, 1^l, \beta_{l+1}, \cdots, \beta_n)=\beta\end{aligned}$$

where all Schubert indices in the above are primitive[3)]. Since γ is a primitive Schubert index of genus $n-l$ and of weight $n-l$, this case is reduced to the case $\beta_1 \geq 2$. *Q.E.D.*

Let α be a Schubert index of genus g satisfying the semigroup condition, $\{\alpha_1<\cdots<\alpha_n\}$ the minimal set of generators for the semigroup $H=H(\alpha)$ and $\varphi: P=k[X_1, \cdots, X_n] \longrightarrow k[t^r]_{r \in H}$ the k-algebra homomorphism defined by $\varphi(X_i)=t^{\alpha_i}$ for $i=1, \cdots, n$. We set $I=\mathrm{Ker}\ \varphi$, $B=P/I$ and $C=\mathrm{Spec}\ B$. Let T_C^1 be the k-vector space of first order deformations of C and D the B-submodule of $\mathrm{Hom}_B(I/I^2, B)$ generated by the homomorphisms $d_{(1)}, d_{(2)}, \cdots, d_{(n)}$ where $d_{(l)}: I/I^2 \longrightarrow B$ is defined by sending $h+I^2$ to $\partial h/\partial X_l+I$ with $h \in I$ for all $l=1, 2, \cdots, n$. Then we have the exact sequence of k-vector spaces

$$0 \longrightarrow D \longrightarrow \mathrm{Hom}_B(I/I^2, B) \xrightarrow{\Phi} T_C^1 \longrightarrow 0^{5)}.$$

Recall that $\mathrm{Hom}_B(I/I^2, B)$ and D have natural gradings through φ[4),5)]. Hence the above exact sequence defines a natural grading on T_C^1 as follows: for any $\nu \in \mathbf{Z}$, we denote by $T_C^1(\nu)$ the image of the ν-th graded piece of $\mathrm{Hom}_B(I/I^2, B)$ by the homomorphim Φ. Assume that $n=3$. Then by the results of Pinkham[4)] and Schaps[6)] we have

$$\dim \mathscr{E}_\alpha = \dim_k \sum_{\nu<0} T_C^1(\nu) - 1.$$

Hence we see the following:

Proposition 2. *We have* $\dim \mathscr{E}_{(0^4,3,3)} = 10$, *hence the Schubert index* $(0^4, 3, 3)$ *is dimensionally proper.*

Proof. If we set $\alpha = (0^4, 3, 3)$, then $\{5, 6, 7\}$ is the minimal set of generators for $H(\alpha)$. It is easily seen that the ideal I is generated by

$$f_1 = -X_1^4 + X_2 X_3^2,\ f_2 = -X_2^2 + X_1 X_3 \text{ and } f_3 = -X_3^3 + X_1^3 X_2.$$

We denote by (h_1, h_2, h_3) the B-module homomorphism $\theta: I/I^2 \longrightarrow B$ defined by $\theta(f_i + I^2) = h_i + I$ for $i = 1, 2, 3$. Since the generators for the ideal I are the 2 by 2 minors of $\begin{bmatrix} X_2 & X_3 \\ X_3^2 & X_1^3 \\ X_1 & X_2^2 \end{bmatrix}$, $\mathrm{Hom}_B(I/I^2, B)$ is generated as a B-module by the homomorphisms

$$\theta_{11} = (0, X_2, -X_1^3),\ \theta_{12} = (0, -X_1, X_3^2),\ \theta_{21} = (-X_2, 0, X_3),\ \theta_{22} = (X_1, 0, -X_2),$$
$$\theta_{31} = (X_1^3, -X_3, 0) \text{ and } \theta_{32} = (-X_3^2, X_2, 0)^{5)}.$$

Then we have

$$d_1 = -\theta_{31} - 3X_1^2\theta_{22},\ d_2 = -\theta_{32} - \theta_{11} \text{ and } d_3 = -2X_3\theta_{21} - \theta_{12}.$$

Hence the following are a k-basis for $\sum_{\nu<0} T_C^1(\nu)$:

$$\theta_{22} \in T_C^1(-15),\ \theta_{21} \in T_C^1(-14),\ X_1\theta_{22} \in T_C^1(-10),\ X_2\theta_{22} \in T_C^1(-9),\ X_3\theta_{22} \in T_C^1(-8),$$
$$\theta_{12} \in T_C^1(-7),\ \theta_{11} \in T_C^1(-6),\ \theta_{31} \in T_C^1(-5),\ X_1X_2\theta_{22} \in T_C^1(-4),\ X_1X_3\theta_{22} \in T_C^1(-3) \text{ and}$$
$$X_2X_3\theta_{22} \in T_C^1(-2).$$

Therefore we obtain

$$\dim \mathscr{E}_\alpha = \dim_k \sum_{\nu<0} T_C^1(\nu) - 1 = 10. \qquad Q.E.D.$$

For example by Proposition 2 and the proof of Proposition 1 we get the following:

Corollary 1. *Let m and n be twe positive integers with $m \leq n$ and $m+n=g$. Then the Schubert index $(0^{g-2}, m, n)$ is dimensionally proper.*

Since any primitive Schubert index of genus g and of weight $\leq g-1$ is dimensionally proper, we propose the following problem:

Problem. Is any primitive Schubert index of genus g and of weight g dimensionally proper?

I do not even know whether the above index is Weierstrass or not.

§ 2. On Primitive Schubert Indices Which are not Dimensionally Proper

In this section we shall give a lot of examples of primitive Schubert indices which are not dimensionally proper.

Definition. Let $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{g-1})$ be a Schubert index of genus g. Then an element of the set $G=\{\alpha_i+i+1 \mid i=0, 1, \cdots, g-1\}$ is called a *gap* of α. The index α is said to be *semi-Weierstrass* if the number of the set $\{r+s \mid r, s\in G\}$ is less than $3g-2$. It is known that any Weierstrass Schubert index is semi-Weierstrass[1].

Example 1. The following are primitive Schubert indices of genus g which are not semi-Weierstrass[7].

1) $(0^n, n-2m, n-2m+1, \cdots, n-2m+m-2, n-m, n-m)$ $m\geq 3$, $n\geq 4m$ and $g=n+m+1$,

2) $(0^n, n-2m, n-2m, n-2m+2, n-2m+2+1, \cdots, n-2m+2+m-2)$, $m\geq 3$, $n\geq 4m$ and $g=n+m+1$,

3) $(0^n, n-2m, n-2m, n-2m+1, \cdots, n-2m+m-1, n-2m+m-1)$, $n\geq 4m$, $m\geq 3$ and $g=n+m+2$,

4) $(0^n, n-m, \alpha_{n+2}, \alpha_{n+3}, n-3)$, $n\geq 12$, $6\leq m\leq [n/2]$, $n-m\leq \alpha_{n+2}\leq \alpha_{n+3}\leq n-3$, $\#\{a+b \mid a, b\in\{n-m, 1+\alpha_{n+2}, 2+\alpha_{n+3}, n\}\}=10$ and $g=n+4$.

Example 2. For any $n\geq 12$, the Schubert index $\alpha=(0^{n-1}, n-6, n-6, n-4, n-4)$ is primitive and semi-Weierstrass, but it is not dimensionally proper.

Proof. Using Proposition in 7) we see that α is semi-Weierstrass. Assume that α is dimensionally proper. From the result due to Eisenbud-Harris[2] we deduce that $\beta=(0^n, n-6, n-6, n-4, n-3)$ is dimensionally proper, hence it is Weierstrass. But β is not semi-Weierstrass[7]. This is a contradiction. *Q.E.D.*

Lemma 1. *Let n, m and l be positive integers. If $n>m+l-2$, then $\alpha=(0^n, l^m)$ is a semi-Weierstrass primitive Schubert index.*

Proof. The first non-gap (resp. the last gap) of α is equal to $n+1$ (resp. $l+n+m$). Because $n>m+l-2$ we have $2(n+1)>l+n+m$, which implies that α is primitive. If

$m \leq 3$, it follows from Proposition in 7) that α is semi-Weierstrass. Assume that $m \geq 4$. Let G be the set of gaps of α. Then we have

$$G=\{1, 2, \cdots, n, l+n+1, l+n+2, \cdots, l+n+m\}.$$

We set $A=\{r+s-2 \mid r, s \in G\}$. It is easily seen that

$$A=[0, 2n+m+l-2] \cup [2n+2l, 2n+2l+2m-2]$$

where for two integers $p \leq q$ we denote by $[p, q]$ the set of integers r with $p \leq r \leq q$, because

$$n-1+n-1-(n+l)=n-l-2>m+l-2-l-2=m-4 \geq 0.$$

Hence we obtain

$$\begin{aligned}\#A &\leq 2n+m+l-1+2n+2l+2m-2-(2n+2l-1)=2n+3m+l-2\\ &=3(n+m)-3-(n-l-1)<3(n+m)-3-(m-3)<3(n+m)-3,\end{aligned}$$

which implies that α is semi-Weierstrass. *Q.E.D.*

Proposition 3. *Let $n>m$ be positive integers with $(n-m)m+4m-6n-1>0$. Then $\alpha=(0^n, m^{n+1-m})$ is a semi-Weierstrass primitive Schubert index which is not dimensionally proper.*

Proof. We have $n-(n+1-m+m-2)=1>0$. From Lemma 1 we deduce that α is semi-Weierstrass and primitive. Now the genus (resp. the weight) of α is equal to $g=2n+1-m$ (resp. $w=m(n+1-m)$). Hence we obtain

$$3g-2-w=3(2n+1-m)-2-m(n+1-m)=-\{(n-m)m+4m-6n-1\}<0,$$

which implies that α must be not dimensionally proper. *Q.E.D.*

By Proposition 3 we get the following examples whose latter one is due to Eisenbud-Harris[2)].

Example 3. The following are semi-Weierstrass primitive Schubert indices which are not dimensionally proper.

(1) $(0^n, 7^{n-6})$, $n \geq 23$, (2) $(0^{3l}, (2l+1)^l)$, $l \geq 6$.

The question of which primitive Schubert indices α are dimensionally proper remains open. For example we would like to know:

Problem. (1) Is any primitive Schubert index of genus g and of weight w with $g \leq w \leq g+12$ dimensionally proper? In a weaker sense is it Weierstrass?

(2) Does there exist a primitive Weierstrass Schubert index which is not dimensionally proper?

§ 3. On Primitive Schubert Indices of Genus $g \leq 8$.

In the last section we investigate which primitive Schubert indices of genus $g \leq 8$ are dimensionally proper (resp. Weierstrass). Since all primitive Schubert indices of weight $\leq g-1$ are dimensionally proper, we give the primitive Schubert indices of weight $\geq g$ in the case $g \leq 8$.

Example 4. (1) The primitive Schubert index of genus 6 and of weight ≥ 6 is $(0^4, 3^2)$.

(2) The primitive Schubert indices of genus 7 and of weight ≥ 7 are $(0^5, 3, 4)$ and $(0^5, 4^2)$.

(3) The primitive Schubert indices of genus 8 and of weight ≥ 8 are $(0^5, 2, 3, 3)$, $(0^6, 4^2)$, $(0^6, 3, 5)$, $(0^5, 3^3)$ and $(0^6, 4, 5)$.

By Corollary 1 we see the following:

Remark. The indices $(0^4, 3^2)$, $(0^5, 3, 4)$, $(0^6, 4^2)$ and $(0^6, 3, 5)$ are dimensionally proper.

Proposition 4. *The indices* $(0^5, 4^2)$, $(0^5, 3^3)$ *and* $(0^6, 4, 5)$ *are Weierstrass.*

Proof. Let $\alpha = \langle 6, 7, 8, 9 \rangle$. Then we have $H(\alpha) = \langle 6, 7, 8, 9 \rangle$ where for any positive integers $a_1, \cdots, a_n$, $\langle a_1, \cdots, a_n \rangle$ denotes the subsemigroup of $\boldsymbol{N}$ generated by $a_1, \cdots, a_n$. Let C be a curve defined by an equation of the form $y^6 = (x-a)(x-b)(x-c)(x-d)^4$ where a, b, c and d are distinct elements of k. Let $f: C \longrightarrow \mathbf{P}^1$ be the surjective morphism defined by sending any point P of C to $(1, x(P))$. We set $f^{-1}((0, 1)) = \{P_\infty\}$. Then $\alpha = \alpha(P_\infty)$[8], which implies that α is Weierstrass.

Let $\alpha = (0^5, 3^3)$. Then we have $H(\alpha) = \langle 6, 7, 8, 17 \rangle$. Since $H(\alpha)$ is 1-neat, α is Weierstrass[9].

Let $\alpha = (0^6, 4, 5)$. Then we have $H = H(\alpha) = \langle 7, 8, 9, 10, 12 \rangle$. Let $\varphi: k[X] = k[X_1, X_2, \cdots, X_5] \longrightarrow k[t^a]_{a \in H}$ be the k-algebra homomorphism defined by $\varphi(X_i) = t^{a_i}$ for $i = 1, \cdots, 5$, where we set $a_1 = 7$, $a_2 = 8$, $a_3 = 9$, $a_4 = 10$ and $a_5 = 12$. It is seen that the ideal Ker φ is generatred by

$$X_1^3 - X_3X_5,\ X_2^2 - X_1X_3,\ X_3^2 - X_2X_4,\ X_4^2 - X_2X_5,\ X_5^2 - X_1^2X_4,\ X_1^2X_2 - X_4X_5,$$
$$X_1X_4 - X_2X_3 \text{ and } X_1X_5 - X_3X_4.$$

Let S be the subsemigroup of $\mathbf{Z}^6$ generated by

$$b_i = e_i \text{ for } i = 1, \cdots, 6,\ b_7 = e_1 + e_2 - e_3,\ b_8 = e_4 + e_5 - e_2,\ b_9 = e_3 + e_5 - e_2 \text{ and}$$
$$b_{10} = e_3 + e_6 - e_2,$$

where for any $1 \leq i \leq 6$ e_i denotes the vector whose i-th component is equal to 1 and whose j-th component is equal to 0 if $j \neq i$. We set

$$g_1 = X_1^2,\ g_2 = X_1,\ g_3 = X_3,\ g_4 = X_2,\ g_5 = X_2,\ g_6 = X_4,\ g_7 = X_5,\ g_8 = X_3,\ g_9 = X_4 \textit{ and } g_{10} = X_5.$$

Let $\pi: k[Y] = k[Y_1, \cdots, Y_{10}] \longrightarrow k[T^s]_{s \in S}$ (resp. $\eta: k[Y] \longrightarrow k[X]$) be the k-algebra homomorphism defined by $\pi(Y_i)=T^{b_i}$ (resp. $\eta(Y_i)=g_i$). Then we see that the ideal Ker φ is generated by the elements of $\eta(\mathrm{Ker}\ \pi)$. Hence α is Weierstrass[10]. *Q.E.D.*

It seems to me that the following questions remain open.

Problem. (1) Is the index $(0^5, 2, 3, 3)$ Weierstrass? Moreover, is it dimensionally proper?

(2) Are the indices $(0^5, 4^2)$, $(0^5, 3^3)$ and $(0^6, 4, 5)$ dimensionally proper?

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