2-D Quantum Gravity

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Abstract

Recently, Knizhnik, Polyakov and Zamolodchikov (KPZ) suggested that 2-D gravity should be quantized in the light cone gauge as described by the trace anomaly action. The derived equation determines all critical exponents of matter fields coupled to 2-D quantum gravity which is described in terms of an SL (2, R) Kac-Moody algebra. In fact, this SL (2, R) Kac-Moody symmetry which they (KPZ) found allowed us to solve the theory completely, at least in the limit of "weak gravity", $1>d$, central charge.

1. Introduction

In recent papers, 2-D quantum gravity has been investigated in the light cone gauge by Polyakov[1] and his group[2]. They discussed the quantization of 2-D induced gravity action. The conformal invariance[3,4] was used to obtain information about the operator products and consequently about the correlation functions of the theory. They discovered a connection with SL (2, R) Virasoro-Kac-Moody algebra. The action which they considered arises in any conformal field theory model coupled to gravity upon integrating out the matter field vacuum fluctuations. Moreover, the understanding of this system is essential for the quantization of strings in noncritical dimensions. It is also related to the theory of random surfaces in statistical mechanics.

After their success of the light cone gauge approach, David[5] and Distler and Kawai[6] proposed a derivation of the gravitational anomalous dimensions from the conformal approach, treating the Liouville exponential interaction as a marginal deformation of the free action.

In the present paper, 2-D quantum gravity in the light cone gauge will be mainly reviewed.

2-D gravitational physics[7] carries historically an interesting structure and has attracted a growing research. As it is well-known, in two dimensions, the usual Einstein action:

$$\int d^2x\sqrt{-g}R, \tag{1}$$

is a topological invariant and therefore has no dynamical contents. Since the two-dimensional Richi tensor, $R_{\mu\nu}$, is identically equal to $1/2\, g_{\mu\nu}R$, the Einstein equation in the vacuum:

$$R_{\mu\nu}-1/2\; g_{\mu\nu}R-\Lambda g_{\mu\nu}=0, \tag{2}$$

has only the non-physical solution $g_{\mu\nu}=0$ when the cosmological constant Λ is nonzero. For

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a vanishing Λ, $g_{\mu\nu}$ is undetermined and the geometry of the two-dimensional space cannot be described by (2). Jackiw and Teitelboim suggested an analogue of the vacuum Einstein equations where space-time geometry is dictated by Liouville dynamics in the form:

$$R-2\Lambda=0. \tag{3}$$

This equation may be obtained from varying the action:

$$S_{JT}=\int d^2x\sqrt{-g}\,(R-2\Lambda)N(x), \tag{4}$$

here $N(x)$ is a scalar field as the role of a Lagrange multiplier. From this action we can get (3) while the variation with respect to the metric $g_{\mu\nu}$ yields

$$(g_{ab}\nabla^{r}\nabla_{r}-\nabla_{a}\nabla_{b})N(x)+\Lambda g_{ab}N(x)=0. \tag{5}$$

Taking the trace of this equation we get the Klein-Gordon equation in de-Sitter space. It is important to note that (5) does not put any further constrains on the metric. At this stage the theory described by the action (4) can be quantized in a canonical manner. The Hamiltonian derived from (4) is a linear combination of the above mentioned constrains (3) and (5) expressed in terms of the canonical variables.

Recently, in 2-D gravity theory, the induced gravity action was used by Polyakov:

$$S_P=\int d^2x\sqrt{g}\,(R\Box^{-1}R+2\Lambda). \tag{6}$$

In two dimensions, the usual Einstein gravity does not exist since the Einstein tensor is identically zero. However, interaction of the metric with the matter fields induces non-trivial equations of motion for the gravitational field ϕ. In this way, using a path-integral approach, a non-local effective action for $g_{\alpha\beta}$ can be obtained. Only in conformal gauge, with line element,

$$\begin{aligned} &ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}=e^{\beta\phi(x)}dx^{+}dx^{+}, \\ &(x^{\pm}=x^{0}\pm x^{1}), \end{aligned} \tag{7}$$

it becomes local, producing a Liouville-type dynamics for the field ϕ. In the Polyakov path integral approach to the quantization of strings, the world sheet metric $h_{\alpha\beta}$ is quantized in the conformal gauge. There the conformal factor of the metric is the only dynamical degree of freedom and it decouples in the critical dimension. On the other hand, quantizing in sub-critical dimensions requires the solution of a non-trivial quantum gravity theory on the world sheet.

2. Light cone gauge

In order to regularize the action we have to introduce a regulator which preserves general covariance. In the conformal gauge this turns out to be difficult. However, recently, the effective action has been studied in a different gauge called the light cone gauge. In this gauge, one makes the theory convergent by adding the covariant regulator M^{-2}

$\int d^2x\sqrt{-h}\,R^2$ to the action. R here is scalar curvature, and M is a cut-off mass.

The action in (6) is changed into the following form:

$$S=d/96\pi\cdot\int d^2x\sqrt{-h}\,[R\Box^{-1}R+R^2/M^2+\Lambda]. \tag{8}$$

Here $\Box$ is the scalar laplacian constructed from $h_{\alpha\beta}$. The first term is the effective action of any two-dimensional coupling to gravity. Here d is equal to the central charge of the Kac-Moody algebra, and d^{-1} will play the role of coupling constant.

In this paper, we use the light cone gauge with the metric:

$$ds^2=h_{\alpha\beta}dx^\alpha dx^\beta=dx^+dx^-+h_{++}(x^+,\,x^-)dx^+dx^+, \tag{9}$$

$$h_{\alpha\beta}=\begin{bmatrix}0 & 1/2\\ 1/2 & h_{++}\end{bmatrix}. \tag{10}$$

If we denote the invariant length by $h_{\alpha\beta}dx^\alpha dx^\beta$ where $\alpha,\,\beta=+,\,-$, by using $h^{\alpha\beta}$ from (10), the non-vanishing Christoffel connections are

$$\Gamma^+_{++}=-\partial_-h_{++},\ \Gamma^-_{++}=\partial_+h_{++}+2h_{++}\partial_-h_{++},\\ \Gamma^-_{+-}=\partial_-h_{++}. \tag{11}$$

Since the regulator term does not affect the analysis in the light cone gauge, now we shall omit this term from the action (8). In order to find the action and equations of motion in this gauge, we denote the following relations, (12)-(14), from the general relations at the case of 4 dimensions by using $T^{++}=4T_{--}$, and the trace anomaly equation [Appendix A], $T^\alpha_\alpha=(d/24\pi)R$. We obtain[8~10)]

$$\delta S=\int T_{--}\delta h_{++}\sqrt{-h}\,d^2x. \tag{12}$$

The curvature scalar is evaluated to be

$$R=4\partial_-^2h_{++}. \tag{13}$$

We obtain the covariant derivative of the induced energy momentum tensor:

$$\nabla_+T_{--}=\partial_+T_{--}-h_{++}\partial_-T_{--}-(\partial_-h_{++})T_{--}=d/24\pi\cdot\partial_-R. \tag{14}$$

To evaluate a gravitational analogue of Wess-Zumino-Novikov-Witten action (WZNW), we first determine $\Box^{-1}R$ in the light cone gauge. To start with, we need to evaluate the action of the laplacian:

$$\Box=h^{\alpha\beta}\nabla_\alpha\nabla_\beta=4(\partial_-)\cdot(\partial_+-h_{++}\partial_-). \tag{15}$$

The action of $\Box^{-1}$ on the scalar curvature is given by

$$D=\Box^{-1}R=(\partial_+-h_{++}\partial_-)^{-1}\partial_-h_{++}. \tag{16}$$

Moreover, h_{++} must be replaced by a scalar field $f(x^+,\,x^-)$ related to h_{++} through the equation:

$$\partial_+ f(x^+, x^-) - h_{++}\partial_- f(x^+, x^-) = 0. \tag{17}$$

In stead of $S[h_{++}]$, here we obtain $S[f]$:

$$S[f] = d/24\pi \cdot \int d^2x \left\{ \frac{\partial_-^2 f \partial_+ \partial_- f}{(\partial_- f)^2} - \frac{(\partial_-^2 f)(\partial_+ f)}{(\partial_- f)^3} \right\}. \tag{18}$$

The equations of motion which satisfy (18) can be obtained from the anomaly relation, (13) and (14). We can easily denote straight

$$\nabla_+ T_{--} = -c/24\pi \cdot \partial_- R = -c/24\pi \cdot \partial_-^3 h_{++}, \quad (c = 26 - d). \tag{19}$$

When the theory is quantized in this gauge, the effect of replacing h_{++} by results in determinants changed $-d$ to $c = 26 - d$. The result of (19) shows that the equation of motion for h_{++} is simply

$$\partial_-^3 h_{++} = 0. \tag{20}$$

Actually by using (16) and $T^{\alpha\beta}$[10] we can also get (20), [Appendix B].

3. Ward identities

The equation of motion, (20), is the starting point for the derivation of Ward identities associated with the residual gauge transformations leaving the form of the metric in (9) invariant. The Ward identities define correlation functions in the theory. For this purpose, we denote gauge variation of h_{++}. ϵ_+ is a parameter for diffeomorphism transformation:

$$\begin{aligned} \delta f &= \epsilon_+ \partial_- f (= \epsilon_+ (\partial/\partial x^-) f(x^+, x^-)), \\ \delta h_{++} &= \nabla_+ \epsilon_+ = (\partial_+ - h_{++}\partial_- + \partial_- h_{++}). \end{aligned} \tag{21}$$

As the result, we find Ward identities by using (19):

$$\begin{aligned} & c/24i\pi \cdot \partial_-^3 \langle h_{++}(z) h_{++}(x_1) \ldots h_{++}(x_N) \rangle \\ & = \sum_{j=1}^{N} \partial_+ \delta(z - x_j) \langle h_{++}(z) h_{++}(x_1) \ldots h_{++}(x_j) \ldots h_{++}(x_N) \rangle \\ & \quad + \sum_{j=1}^{N} [\delta(z - x_j) \partial/\partial x_j^- - \partial_- \delta(z - x_j)] \langle h_{++}(x_1) \ldots h_{++}(x_N) \rangle. \end{aligned} \tag{22}$$

Moreover, the Word identities are then integrated to yield the recursive correlation functions of an arbitrary number of h_{++}. By means of the convenient rescaling $h_{++} \to (c/6) h_{++}$ and the following relation:

$$1/4i\pi \cdot \partial_-^3 \{(z^-)^2/z^+\} = \delta^{(2)}(z). \tag{23}$$

We obtain the correlation functions in the theory:

$$\begin{aligned} & \langle h_{++}(z) h_{++}(x_1) \ldots h_{++}(x_N) \rangle \\ & = -c/6 \cdot \sum_j \{(z^- - x_j^-)^2/(z^+ - x_j^+)^2\} \langle h_{++}(x_1) \ldots h_{++}(x_j) \ldots h_{++}(x_N) \rangle \\ & \quad + \sum_j [\{(z^- - x_j^-)^2/(z^+ - x_j^+)\}(\partial/\partial x_j^-) + 2(z^- - x_j)/(z^+ - x_j^+)] \langle h_{++}(x) \ldots h_{++}(x_N) \rangle. \end{aligned} \tag{24}$$

Furthermore, we also can give relations for the arbitrary primary fields, ϕ, with transformation laws under the change of x^- coordinate:

$$\delta\phi = \epsilon_+\partial_-\phi + \lambda(\partial_-\epsilon_+)\phi. \tag{25}$$

Now we have

$$\langle h_{++}(z)\phi(x_1)\ \dots\ \phi(x_N)\rangle = \sum_j \left[\frac{z^- - x_j^-}{z^+ - x_j^+}\frac{\partial}{\partial x_j^-} + 2\lambda\frac{z^- - x_j^-}{z^+ - x_j^+}\right]\langle\phi(x_1)\ \dots\ \phi(x_N)\rangle, \tag{26}$$

where λ is the parameter related to SL(2, R) spin.

4. Emergence of SL(2, *R*) symmetry

Parametrizing the most general solution of (20) as

$$h_{++}(x^+, x^-) = J^+(x^+) - 2J^0(x^+)x^- + J^-(x^+)(x^-)^2, \tag{27}$$

where $J^a(x^+)$ $(a=0, +, -)$ are the currents of an SL(2, R) current algebra, Kac-Moody algebra. Here we use the following generators of SL(2, R), L_j^a $(a=0, +, -)$ defined by

$$\begin{aligned} &L_j^+ = (x_j^-)^2\partial_- + 2\lambda x_j^-, \\ &L_j^0 = x_j^-\partial_- + \lambda, \\ &L_j^- = \partial_-, \\ &(\partial_- = \partial/\partial x_j^-), \end{aligned} \tag{28}$$

satisfy an SL(2, R) Lie algebra:

$$\begin{aligned} &[L_j^a, L_j^b] = -f_j^{ab}L_j^c, \\ &\quad f_0^{+-} = 2,\ f_+^{0+} = -1,\ f_-^{0-} = +1. \end{aligned} \tag{29}$$

The operator product expansion of two currents [11] is given by

$$J^a(x^+)J^b(x^+) = -K/2\cdot[\eta_{ab}/(x^+ - x'^+)^2] + f_c^{ab}[J^c(x'^+)/(x^+ - x'^+)] + \dots, \tag{30}$$

where η_{ab} is the Killing metric tensor for SL(2, R) $(\eta_{00} = -1,\ \eta_{-+} = \eta_{+-} = 1/2)$ and f_c^{ab} are the structure constants. By substituting (27) into (26) we fied

$$\langle J^a(z)\phi(x_1)\ \dots\ \phi(x_N)\rangle = \sum_j L_j^a/(z - x_j^+)\cdot\langle\phi(x_1)\ \dots\ \phi(x_N)\rangle. \tag{31}$$

Using the Sugawara construction[12) [Appendix C]:

$$T_{++}^{\mathrm{gravity}} = 1/(K+2)\cdot\eta_{ab}J^aJ^b + \partial_+J^0, \tag{32}$$

where K is the central charge of the SL(2, R) current algebra, we find the following differential equation of the correlation function involving primary fields ϕ:

$$-(K+2)\sum_j \partial_+^j\langle\phi(x_1)\ \dots\ \phi(x_N)\rangle = \sum_{j\neq k}\eta_{ab}L_j^aL_k^b/(x_j^+ - x_k^+)\cdot\langle\phi(x_1)\ \dots\ \phi(x_N)\rangle. \tag{33}$$

L_j^a are the differential operators representing the action of the current $J^a(x_j^+)$ on the space of primary fields.

5. Total energy-momentum tensor

We resolve some difficulties with renormalization and find explicit formula for the spectrum of anomalous dimensions caused by the fluctuations of intrinsic geometry of random surface. We assume the few families of Majorana fermions interacting with the 2-D gravity. We generally use the following Lagrangian:

$$L=\Psi_-(\partial_+-h_{++}\partial_-)\Psi_-+\Psi_+(\partial_--h_{--}\partial_+)\Psi_+. \tag{34}$$

In order to treat the gravitational field, we fix $h_{--}=0$, and integrate over fermions and a gravitational analogue of WZNW action, $S(h_{++})$. A remarkable property of this action was previously SL(2, R) current algebra, generated by h_{++}, which supplies differential equations defining correlation functions of theory. These differential equations involve constant parameters which are subjected to finite renormalization. We solve the problem of computing this effect. By adding the covariant regulator to the action we make the theory convergent:

$$S_{\mathrm{reg}}=\int(R^2/M^2)\sqrt{-h}\,d^2x=4M^{-2}\int(\partial_-^2h_{++})\sqrt{-h}\,d^2x. \tag{35}$$

This term modifies the propagator of h-field without touching vertices. Here we consider the improved Lagrangian including the contributions from ghost fields and gravity system:

$$L(\mathrm{total})=L(\mathrm{matter})+L(\mathrm{ghost})+L(\mathrm{gravity}), \tag{36}$$

$$L(\mathrm{matter})=\Psi_-(\partial_+-h_{++}\partial_-)\Psi_-+\Psi_+(\partial_--h_{--}\partial_+)\Psi_+, \tag{37}$$

$$L(\mathrm{ghost})=\eta_{++}\nabla_-\xi_-+(\nabla_+\xi_-+\nabla_-\xi_+), \tag{38}$$

where we introduce the pair of ghost (ξ_+, ξ_-) and anti-ghost (η_{++}, ζ). The gravity term in the Lagrangian is certain local functional of (h_{++}, h_{--}, h_{+-}) guaranteeing general covariance. Furthermore, we need change the previous gauge (10) into the following gauge:

$$h_{--}=h_{--}(x),\ h_{+-}=h_{+-}(x), \tag{39}$$

where $h_{--}(x)$ and $h_{+-}(x)$ are fixed as certain functions.

Here we consider the total action including matter, ghost fields and gravity system:

$$S(\mathrm{total})=S(\mathrm{matter})+S(\mathrm{ghost})+S(\mathrm{gravity}), \tag{40}$$

$$S(\mathrm{gravity})=S(h_{++},\ h_{--},\ h_{+-}). \tag{41}$$

The total action is invariant under diffeomorphisms. The gravity action is functional of (h_{++}, h_{--}, h_{+-}) the same as Lagrangian. Hence the variation of the total action, S tot, and also any gauge invariant quantity must be independent of $h_{--}(x)$ and $h_{+-}(x)$ and we have the condition:

$$\begin{aligned}\delta S\,\mathrm{tot}/\delta h_{--}(x)\,|_{h_{--}=0}&=T_{++}\,\mathrm{tot}=0,\\ \delta S\,\mathrm{tot}/\delta h_{+-}(x)\,|_{h_{+-}=0}&=T_{+-}\,\mathrm{tot}=0.\end{aligned} \tag{42}$$

These variations are components of the total energy-momentum tensor, T_{++} tot, and imply the vanishing of the central charge of the Virasoro algebra generated by T_{++} tot[1,2].

According to (42), we get the following relation:

$$S(\text{gravity}) \approx \int d^2x \{h_{--} T_{++}\text{gra}(h_{++}) + h_{+-}\theta(h_{++})\}, \tag{43}$$

where T_{++} gra is the energy-momentum tensor of the gravity system. Now, in order to find T_{++} tot including the matter, ghosts and gravity system, we consider the invariance under transformations:

$$\delta h_{++} = 2\nabla_+\xi_+,\ \delta h_{+-} = \nabla_+\xi_- + \nabla_-\xi_+,\ \delta h_{--} = 2\nabla_-\xi_-. \tag{44}$$

We find

$$T_{++}\text{gra} \propto 1/2 \cdot [(\partial_- h_{++})^2 - 2h_{++}\partial_-^2 h_{++}] + \partial_+\partial_+ h_{++},\ \theta \propto \partial_-^2 h_{++}. \tag{45}$$

We get the total energy-momentum tensor by using the condition (42), Lagrangian (37), (38) and (45):

$$T_{++}\text{tot} = T_{++}\text{matt} + T_{++}\text{gho} + T_{++}\text{gra},\ \theta(h_{++}) = 0, \tag{46}$$

and so,

$$\begin{aligned} T_{++}\text{matt} &= \Psi_+\partial_+\Psi_+, \\ T_{++}\text{gho} &= \eta_{++}\partial_+\xi_- + \partial_+\xi_+, \\ T_{++}\text{gra} &= 1/(K+2)\cdot\eta_{ab}J^aJ^b + \partial_+J^0, \end{aligned} \tag{47}$$

where K is the central charge of the SL(2, R) current algebra. T_{++} gra is actually derived straight from Sugawara energy-momentum tensor (C-2) [Appendix C] with use of (45). T_{++} tot satisfies a Virasoro algebra.

6. Total central charges

This condition (46) implies a relation for the central charges. Moreover, the total central charge of the Virasoro algebra generated by T_{++} tot must vanish for consistency with use of (43):

$$\begin{aligned} C\text{tot} &= C\text{matt} + C\text{gho} + C\text{gra} = 0, \\ C\text{matt} &= C(\Psi) = d, \\ C\text{gho} &= C(\eta_{++},\ \xi_-) + C(\zeta,\ \xi_+) = -26 - 2, \\ C\text{gra} &= 3K/(K+2) - 6K\ \text{[Appendix C]}, \end{aligned} \tag{49}$$

and so,

$$C\text{tot} = d - 28 + 3K/(K+2) - 6K = 0. \tag{50}$$

Note that 1), 2) sometimes use K for the 'mirror charge': $K \to -K-4$. We denote the 'mirror charge' by K. We can express it in terms of material central charge:

$$d - 13 = -6(K+2) - 6/(K+2). \tag{51}$$

Because of renormalization effects coming from gravitational interactions, the physical value of K, defined as a central charge of current algebra, is given by (51). Note that the above argument in deriving (51) does not depend on the specific form (34) of matter Lagrangian. The SL(2, R) current symmetry and the equation (51) determining its central charge K stand for the general case of gravity induced by the massless matter with Virasoro central charge d.

7. Conclusion

KPZ[1,2] derived the KPZ equation which determines all critical exponents of matter fields chirally coupled to quantum gravity:

$$\Delta-\Delta(0)=\Delta(1-\Delta)/(K+2), \tag{52}$$

where Δ and $\Delta(0)$ are the conformal dimensions of a primary spinless ϕ_λ field with and without induced gravity respectively. Moreover, this KPZ equation (52) determines the SL(2, R) weight $\lambda=-\Delta$ of any ϕ_λ interacting with the induced gravity. The quantity of Δ represents the scaling dimension of ϕ_λ in the presence of gravity.

According to the relations (51) and (52), we can compute critical exponents for massless field theory interacting with the induced gravity in the "weak gravity" regime $d\leq 1$ or $d\geq 25$. In this region, the quadratic equation (51) possesses real solutions for K. We obtain the solution:

$$K+3=1/12\{d-1-[(1-d)(25-d)]^{1/2}\}. \tag{53}$$

Generally we can examine the quantum gravity induced by any "minimal" conformal field theory with Virasoro central charge $d<1$ and the dimensions $\Delta(0)$ given by Kac spectrum[3]. The KPZ equation (52) shows in this case that the "gravitational dressing" converts the degenerate Virasoro dimensions $\Delta(0)$ nm into the values:

$$\Delta nm=-(1+K)/2+[(K+2)n/2-m/2], \tag{54}$$

which correspond to the degenerate representation of SL(2, R) Kac-Moody algebra.

Nevertheless, at the most physically interesting region $d>1$, the complex values are given for the central charge K and exponents. The "strong gravity" at the region $d>1$ is an unsolved problem.

Recently it has been also investigated that minimal models coupled to 2-D gravity have special type of resonant correlations defined by the free field representations and are computable[16]. Moreover, in order to treat the problem of the breakdown of Polyakov's chiral light cone gauge in the strong coupling regime, $1<d$ matter <25, it is also argued that the appearance of spiky structures is shown to be related to the anticonformal limit of quasiconformal mappings, the so called Beltrami parametrization[17].

Appendix A

In the formalism by using a semiclassical approach to gravity, the gravitational field is treated as a classical background on which the various matter quantum fields propagate;

moreover, we assume that the expectation value in a suitable chosen state of the matter energy-momentum tensor operator, $\langle T_{\mu\nu}\rangle$, acts as a source for the evolution of the background space-time. Here we consider the semiclassical Einstein equations (units are such that, $G=c=\hbar=1$):

$$G_{\mu\nu}+g_{\mu\nu}=8\pi\langle T_{\mu\nu}\rangle, \tag{A-1}$$

where $G_{\mu\nu}$ is the Einstein tensor. In two-dimensions, $G_{\mu\nu}\equiv 0$, as already observed, while $\langle T_{\mu\nu}\rangle$, generally a non-local geometrical object, can be explicitly obtained by using the following properties, $\langle T_{\mu\nu}\rangle$ is covariantly conserved:

$$\nabla^{\nu}\langle T_{\mu\nu}\rangle=0, \tag{A-2}$$

and has trace given by the trace anomaly[8]:

$$\langle T^{\mu}_{\mu}\rangle=-\frac{a}{24\pi}R, \tag{A-3}$$

where R is the scalar curvature and "a" is a constant determined by the number and species of fields entering the theory.

Appendix B

The problem[10] of finding (16) reduces to finding a function D such that

$$(\partial_{+}-h_{++}\partial_{-})=\partial_{-}h_{++}. \tag{B-1}$$

To find a solution of this non-trivial equation, we first use an auxiliary field $f(x^{+}, x^{-})$ in (17) that satisfies a homogeneous form of (B-1). The induced energy momentum tensor is given by the functional derivative:

$$T^{\alpha\beta}(x)=-2/\sqrt{-h}\cdot(\delta S/\delta h_{\alpha\beta}(x)). \tag{B-2}$$

Standard but lengthy manipulations give

$$T^{\alpha\beta}=-d/48\pi\cdot[2\nabla^{\alpha}\nabla^{\beta}D-\nabla^{\alpha}D\nabla^{\beta}D-h^{\alpha\beta}(2R-1/2\nabla_{\gamma}D\nabla^{\gamma}D-1/2\Lambda)], \tag{B-3}$$

where our conventions are $R^{\delta}{}_{\gamma\alpha\beta}=\partial_{\alpha}\Gamma^{\delta}_{\beta\gamma}+\Gamma^{\delta}_{\alpha\rho}\Gamma^{\delta}_{\beta\gamma}-(\alpha\leftrightarrow\beta)$, $R=h^{\beta\gamma}R^{\alpha}_{\gamma\alpha\beta}$, $D=\Box^{-1}R$.

By using (16), (B-1) and (B-3) we can immediately find

$$T_{--}=-d/48\pi\cdot[2\partial_{-}^{2}D-(\partial_{-}D)^{2}], \tag{B-4}$$

and since $T^{++}=4T_{--}$ is the variation with respect to the dynamical variable h_{++}, setting it to zero will give the operator equation of motion:

$$2\partial_{-}^{2}D=(\partial_{-}D)^{2}. \tag{B-5}$$

Differentiating (B-5) with respect to ∂_{+} and using the definition of (B-1) for D, we get

$$\partial_{-}^{3}h_{++}=0 \tag{B-6}$$

This equation is the same as (20) obtained previously.

Appendix C

Now we determine the central charge of the T_{++}gra. We can decompose h_{++} according to the $J^a(x^+)$ generating SL(2, R) current algebra:

$$h_{++}=J^+-2J^0x^-+J^-(x^-)^2. \tag{C-1}$$

With the use of the Sugawara energy-momentum tensor [12] and h_{++} (C-1), we obtain

$$T_{++}(\mathrm{sug})=1/(K+2)\cdot\eta_{ab}J^aJ^b=1/(K+2)\cdot\{1/2\cdot[(\partial_-h_{++})^2-2h_{++}\partial_-^2h_{++}]\}. \tag{C-2}$$

We find T_{++}gra, the Sugawara construction (32), in terms of current, by using (45) and (46):

$$T_{++}\mathrm{gra}=1/(K+2)\cdot\eta_{ab}J^aJ^b+\partial_+J^0. \tag{C-3}$$

The central charge, Cgra, for this energy-momentum tensor is obtained[11]:

$$C\mathrm{gra}=K\dim g/(K+Cg)-6K=3K/(K+2)-6K. \tag{C-4}$$

Here dim $g=\delta^{aa}$ is the dimension of the group G. Cg is defined as $f^{acb}f^{bcd}=Cg\delta^{ab}$, where f^{abc} are the structure constants of Lie algebra of the group G.

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