

# On the Existence of Weierstrass Points in the Lower Genus Cases

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## Abstract

Let  $H$  be a numerical semigroup, i.e., a subsemigroup of the additive semigroup  $N$  of the non-negative integers whose complement  $N \setminus H$  in  $N$  is finite. Then the order of the set  $N \setminus H$  is called the *genus* of  $H$ . If twice the smallest positive integer of  $H$  is larger than the largest integer of  $N \setminus H$ , we say that  $H$  is *primitive*. It is known that any numerical semigroup  $H$  of genus  $\leq 6$  is Weierstrass, i.e., there exists a pointed complete non-singular irreducible algebraic curve  $(C, P)$  over an algebraically closed field  $k$  of characteristic 0 such that  $H$  is the set of integers which are pole orders at  $P$  of regular functions on  $C \setminus \{P\}$ . In this paper we show that any numerical semigroup of genus 7 (resp. any primitive numerical semigroup of genus 8) is Weierstrass.

## §1. On numerical semigroups of genus 7

For a numerical semigroup  $H$  of genus  $g$ ,  $M(H)$  denotes the minimal set of generators for  $H$ . Let  $N \setminus H = \{h_0 < h_1 < \cdots < h_{g-1}\}$ . Then we set

$$\alpha(H) = (\alpha_0(H), \alpha_1(H), \dots, \alpha_{g-1}(H)),$$

where  $\alpha_i(H) = h_i - i - 1$  for any  $i = 0, 1, \dots, g-1$ . We set  $w(H) = \sum_{i=0}^{g-1} \alpha_i(H)$ , which is called the *weight* of  $H$ . Then the following table shows all numerical semigroups  $H$  of genus 7, where  $P$  (resp.  $N$ ) means that  $H$  is primitive (resp. non-primitive).

	$N/H$	$M(H)$	$\alpha(H)$	$w(H)$
(1)	$\{1, 3, 5, 7, 9, 11, 13\}$	$\{2, 15\}$	$N \quad (0, 1, 2, 3, 4, 5, 6)$	21
(2)	$\{1, 2, 4, 5, 7, 10, 13\}$	$\{3, 8\}$	$N \quad (0, 0, 1, 1, 2, 4, 6)$	14
(3)	$\{1, 2, 4, 5, 7, 8, 11\}$	$\{3, 10, 14\}$	$N \quad (0, 0, 1, 1, 2, 2, 4)$	10
(4)	$\{1, 2, 4, 5, 7, 8, 10\}$	$\{3, 11, 13\}$	$N \quad (0, 0, 1, 1, 2, 2, 3)$	9
(5)	$\{1, 2, 3, 5, 7, 9, 13\}$	$\{4, 6, 11\}$	$N \quad (0, 0, 0, 1, 2, 3, 6)$	12
(6)	$\{1, 2, 3, 5, 7, 9, 11\}$	$\{4, 6, 13, 15\}$	$N \quad (0, 0, 0, 1, 2, 3, 4)$	10
(7)	$\{1, 2, 3, 5, 6, 9, 13\}$	$\{4, 7, 10\}$	$N \quad (0, 0, 0, 1, 1, 3, 6)$	11
(8)	$\{1, 2, 3, 5, 6, 9, 10\}$	$\{4, 7, 13\}$	$N \quad (0, 0, 0, 1, 1, 3, 3)$	8
(9)	$\{1, 2, 3, 5, 6, 7, 11\}$	$\{4, 9, 10, 15\}$	$N \quad (0, 0, 0, 1, 1, 1, 4)$	7
(10)	$\{1, 2, 3, 5, 6, 7, 10\}$	$\{4, 9, 11, 14\}$	$N \quad (0, 0, 0, 1, 1, 1, 3)$	6
(11)	$\{1, 2, 3, 5, 6, 7, 9\}$	$\{4, 10, 11, 13\}$	$N \quad (0, 0, 0, 1, 1, 1, 2)$	5
(12)	$\{1, 2, 3, 4, 7, 8, 13\}$	$\{5, 6, 9\}$	$N \quad (0^4, 2, 2, 6)$	10
(13)	$\{1, 2, 3, 4, 7, 8, 9\}$	$\{5, 6, 13, 14\}$	$P \quad (0^4, 2, 2, 2)$	6

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(14)	$\{1, 2, 3, 4, 6, 9, 11\}$	$\{5, 7, 8\}$	$N$	$(0^4, 1, 3, 4)$	8
(15)	$\{1, 2, 3, 4, 6, 8, 13\}$	$\{5, 7, 9, 11\}$	$N$	$(0^4, 1, 2, 6)$	9
(16)	$\{1, 2, 3, 4, 6, 8, 11\}$	$\{5, 7, 9, 13\}$	$N$	$(0^4, 1, 2, 4)$	7
(17)	$\{1, 2, 3, 4, 6, 8, 9\}$	$\{5, 7, 11, 13\}$	$P$	$(0^4, 1, 2, 2)$	5
(18)	$\{1, 2, 3, 4, 6, 7, 12\}$	$\{5, 8, 9, 11\}$	$N$	$(0^4, 1, 1, 5)$	7
(19)	$\{1, 2, 3, 4, 6, 7, 11\}$	$\{5, 8, 9, 12\}$	$N$	$(0^4, 1, 1, 4)$	6
(20)	$\{1, 2, 3, 4, 6, 7, 9\}$	$\{5, 8, 11, 12, 14\}$	$P$	$(0^4, 1, 1, 2)$	4
(21)	$\{1, 2, 3, 4, 6, 7, 8\}$	$\{5, 9, 11, 12, 13\}$	$P$	$(0^4, 1, 1, 1)$	3
(22)	$\{1, 2, 3, 4, 5, 10, 11\}$	$\{6, 7, 8, 9\}$	$P$	$(0^5, 4, 4)$	8
(23)	$\{1, 2, 3, 4, 5, 9, 11\}$	$\{6, 7, 8, 10\}$	$P$	$(0^5, 3, 4)$	7
(24)	$\{1, 2, 3, 4, 5, 9, 10\}$	$\{6, 7, 8, 11\}$	$P$	$(0^5, 3, 3)$	6
(25)	$\{1, 2, 3, 4, 5, 8, 11\}$	$\{6, 7, 9, 10\}$	$P$	$(0^5, 2, 4)$	6
(26)	$\{1, 2, 3, 4, 5, 8, 10\}$	$\{6, 7, 9, 11\}$	$P$	$(0^5, 2, 3)$	5
(27)	$\{1, 2, 3, 4, 5, 8, 9\}$	$\{6, 7, 10, 11, 15\}$	$P$	$(0^5, 2, 2)$	4
(28)	$\{1, 2, 3, 4, 5, 7, 13\}$	$\{6, 8, 9, 10, 11\}$	$N$	$(0^5, 1, 6)$	7
(29)	$\{1, 2, 3, 4, 5, 7, 11\}$	$\{6, 8, 9, 10, 13\}$	$P$	$(0^5, 1, 4)$	5
(30)	$\{1, 2, 3, 4, 5, 7, 10\}$	$\{6, 8, 9, 11, 13\}$	$P$	$(0^5, 1, 3)$	4
(31)	$\{1, 2, 3, 4, 5, 7, 9\}$	$\{6, 8, 10, 11, 13, 15\}$	$P$	$(0^5, 1, 2)$	3
(32)	$\{1, 2, 3, 4, 5, 7, 8\}$	$\{6, 9, 10, 11, 13, 14\}$	$P$	$(0^5, 1, 1)$	2
(33)	$\{1, 2, 3, 4, 5, 6, 13\}$	$\{7, 8, 9, 10, 11, 12\}$	$P$	$(0^6, 6)$	6
(34)	$\{1, 2, 3, 4, 5, 6, 12\}$	$\{7, 8, 9, 10, 11, 13\}$	$P$	$(0^6, 5)$	5
(35)	$\{1, 2, 3, 4, 5, 6, 11\}$	$\{7, 8, 9, 10, 12, 13\}$	$P$	$(0^6, 4)$	4
(36)	$\{1, 2, 3, 4, 5, 6, 10\}$	$\{7, 8, 9, 11, 12, 13\}$	$P$	$(0^6, 3)$	3
(37)	$\{1, 2, 3, 4, 5, 6, 9\}$	$\{7, 8, 10, 11, 12, 13\}$	$P$	$(0^6, 2)$	2
(38)	$\{1, 2, 3, 4, 5, 6, 8\}$	$\{7, 9, 10, 11, 12, 13, 15\}$	$P$	$(0^6, 1)$	1
(39)	$\{1, 2, 3, 4, 5, 6, 7\}$	$\{8, 9, 10, 11, 12, 13, 14, 15\}$	$P$	$(0^6, 0)$	0

If the smallest positive integer of  $H$  is less than or equal to 5, then  $H$  is Weierstrass<sup>1-3)</sup>. All primitive numerical semigroups  $H$  of genus  $g=7$  and weight  $\leq g-1=6$  are Weierstrass<sup>4,5)</sup>. Moreover,  $H$  with  $M(H)=\{6, 7, 8, 9\}$  (resp.  $\{6, 7, 8, 10\}$ ) is Weierstrass<sup>6)</sup>. Hence to show that any numerical semigroup of genus 7 is Weierstrass it suffices to show that  $H$  with  $M(H)=\{6, 8, 9, 10, 11\}$  is Weierstrass (See §3).

## §2. On primitive numerical semigroups of genus 8.

The following table shows all primitive numerical semigroups  $H$  of genus 8.

	$N/H$	$M(H)$	$\alpha(H)$	$w(H)$
(1)	$\{1, 2, 3, 4, 6, 7, 8, 9\}$	$\{5, 11, 12, 13, 14\}$	$(0^4, 1^4)$	4
(2)	$\{1, 2, 3, 4, 5, 9, 10, 11\}$	$\{6, 7, 8, 17\}$	$(0^5, 3^3)$	9
(3)	$\{1, 2, 3, 4, 5, 8, 10, 11\}$	$\{6, 7, 9, 17\}$	$(0^5, 2, 3, 3)$	8

(4)	{1, 2, 3, 4, 5, 8, 9, 11}	{6, 7, 10, 15}	$(0^5, 2, 2, 3)$	7
(5)	{1, 2, 3, 4, 5, 8, 9, 10}	{6, 7, 11, 15, 16}	$(0^5, 2^3)$	6
(6)	{1, 2, 3, 4, 5, 7, 10, 11}	{6, 8, 9, 13}	$(0^5, 1, 3, 3)$	7
(7)	{1, 2, 3, 4, 5, 7, 9, 11}	{6, 8, 10, 13, 15, 17}	$(0^5, 1, 2, 3)$	6
(8)	{1, 2, 3, 4, 5, 7, 9, 10}	{6, 8, 11, 13, 15}	$(0^5, 1, 2, 2)$	5
(9)	{1, 2, 3, 4, 5, 7, 8, 11}	{6, 9, 10, 13, 14, 17}	$(0^5, 1, 1, 3)$	5
(10)	{1, 2, 3, 4, 5, 7, 8, 10}	{6, 9, 11, 13, 14, 16}	$(0^5, 1, 1, 2)$	4
(11)	{1, 2, 3, 4, 5, 7, 8, 9}	{6, 10, 11, 13, 14, 15}	$(0^5, 1^3)$	3
(12)	{1, 2, 3, 4, 5, 6, 12, 13}	{7, 8, 9, 10, 11}	$(0^6, 5, 5)$	10
(13)	{1, 2, 3, 4, 5, 6, 11, 13}	{7, 8, 9, 10, 12}	$(0^6, 4, 5)$	9
(14)	{1, 2, 3, 4, 5, 6, 11, 12}	{7, 8, 9, 10, 13}	$(0^6, 4, 4)$	8
(15)	{1, 2, 3, 4, 5, 6, 10, 13}	{7, 8, 9, 11, 12}	$(0^6, 3, 5)$	8
(16)	{1, 2, 3, 4, 5, 6, 10, 12}	{7, 8, 9, 11, 13}	$(0^6, 3, 4)$	7
(17)	{1, 2, 3, 4, 5, 6, 10, 11}	{7, 8, 9, 12, 13}	$(0^6, 3, 3)$	6
(18)	{1, 2, 3, 4, 5, 6, 9, 13}	{7, 8, 10, 11, 12}	$(0^6, 2, 5)$	7
(19)	{1, 2, 3, 4, 5, 6, 9, 12}	{7, 8, 10, 11, 13}	$(0^6, 2, 4)$	6
(20)	{1, 2, 3, 4, 5, 6, 9, 11}	{7, 8, 10, 12, 13}	$(0^6, 2, 3)$	5
(21)	{1, 2, 3, 4, 5, 6, 9, 10}	{7, 8, 11, 12, 13, 17}	$(0^6, 2, 2)$	4
(22)	{1, 2, 3, 4, 5, 6, 8, 13}	{7, 9, 10, 11, 12, 15}	$(0^6, 1, 5)$	6
(23)	{1, 2, 3, 4, 5, 6, 8, 12}	{7, 9, 10, 11, 13, 15}	$(0^6, 1, 4)$	5
(24)	{1, 2, 3, 4, 5, 6, 8, 11}	{7, 9, 10, 12, 13, 15}	$(0^6, 1, 3)$	4
(25)	{1, 2, 3, 4, 5, 6, 8, 10}	{7, 9, 11, 12, 13, 15, 17}	$(0^6, 1, 2)$	3
(26)	{1, 2, 3, 4, 5, 6, 8, 9}	{7, 10, 11, 12, 13, 15, 16}	$(0^6, 1, 1)$	2
(27)	{1, 2, 3, 4, 5, 6, 7, 15}	{8, 9, 10, 11, 12, 13, 14}	$(0^7, 7)$	7
(28)	{1, 2, 3, 4, 5, 6, 7, 14}	{8, 9, 10, 11, 12, 13, 15}	$(0^7, 6)$	6
(29)	{1, 2, 3, 4, 5, 6, 7, 13}	{8, 9, 10, 11, 12, 14, 15}	$(0^7, 5)$	5
(30)	{1, 2, 3, 4, 5, 6, 7, 12}	{8, 9, 10, 11, 13, 14, 15}	$(0^7, 4)$	4
(31)	{1, 2, 3, 4, 5, 6, 7, 11}	{8, 9, 10, 12, 13, 14, 15}	$(0^7, 3)$	3
(32)	{1, 2, 3, 4, 5, 6, 7, 10}	{8, 9, 11, 12, 13, 14, 15}	$(0^7, 2)$	2
(33)	{1, 2, 3, 4, 5, 6, 7, 9}	{8, 10, 11, 12, 13, 14, 15, 17}	$(0^7, 1)$	1
(34)	{1, 2, 3, 4, 5, 6, 7, 8}	{9, 10, 11, 12, 13, 14, 15, 16, 17}	$(0^7, 0)$	0

All primitive numerical semigroups  $H$  of genus  $g=8$  and weight  $\leq g-1=7$  are Weierstrass<sup>4,5)</sup>. Moreover,  $H$  is Weierstrass if  $\alpha(H)$  is one of the following:

$$(0^5, 3^3), (0^6, 4, 5), (0^6, 4, 4), (0^6, 3, 5)^{6)}.$$

Hence to show that any primitive numerical semigroup of genus 8 is Weierstrass it suffices to show that  $H$  with  $M(H)=\{6, 7, 9, 17\}$  (resp.  $M(H)=\{7, 8, 9, 10, 11\}$ ) is Weierstrass (See §4 (resp. §5)).

### §3. On the numerical semigroup generated by 6, 8, 9, 10 and 11

Let  $H$  be the numerical semigroup with  $M(H) = \{6, 8, 9, 10, 11\}$ . In this section we shall show that there exists a pointed curve  $(C, P)$  such that  $H(P) = H$ , where  $H(P)$  is the set of integers which are pole orders at  $P$  of regular functions on  $C \setminus \{P\}$ .

Let  $E$  be an elliptic curve over  $k$  with the origin  $Q'$ . Let  $P'_1$  be a point of  $E$  such that  $P'_1 \neq Q'$  and  $2P'_1 = Q'$ , i.e.,  $2P'_1 \sim 2Q'$ . Moreover,  $P'_2$  denotes a point of  $E$  such that  $P'_2 \neq Q'$  and  $3P'_2 = -4P'_1$ , i.e.,  $4P'_1 + 3P'_2 \sim 7Q'$ . Take  $z \in K(E)$  such that  $\text{div}(z) = 4P'_1 + 3P'_2 - 7Q'$ , where  $K(E)$  denotes the function field of  $E$ . Let  $\pi: C \rightarrow E$  be the surjective morphism corresponding to the inclusion  $K(E) \subset K(E)(z^{1/7}) = K(C)$ . Let  $y \in K(C)$  and  $\sigma \in \text{Aut}(K(C)/K(E))$  such that  $\sigma(y) = \zeta_7 y$  and  $\text{div}_E(y^7) = 4P'_1 + 3P'_2 - 7Q'$ , where  $\zeta_7$  is a primitive 7-th root of unity. Then there are only two ramification points  $P_1$  and  $P_2$  over  $P'_1$  and  $P'_2$  respectively and the ramification indices are 7. Hence by Riemann-Hurwitz formula the genus of  $C$  is 7.

**Theorem 3.1.** *The semigroup  $H(P_2)$  is generated by 6, 8, 9, 10 and 11, i.e.,  $H(P_2) = H$ .*

*Proof.* We have

$$\text{div}(y) = 4P_1 + 3P_2 - \pi^*(Q')$$

and

$$\text{div}(dy) = 3P_1 + 2P_2 - 2\pi^*(Q') + \sum_{i=1}^3 \pi^*(R'_i),$$

where  $R'_i$ 's are points of  $E$  which are distinct from  $P'_1$ ,  $P'_2$  and  $Q'$ . For any  $f \in K(E)$ , we set  $\text{div}_E(f) = \sum_{P' \in E} n(P')P'$ . Then for any  $r \in \mathbb{N}$  we obtain

$$\begin{aligned} \text{div}\left(\frac{f dy}{y^{1-r}}\right) &= \{7n(P'_1) + 3 + 4(r-1)\}P_1 + \{7n(P'_2) + 2 + 3(r-1)\}P_2 + \{n(Q') - r - 1\}\pi^*(Q') \\ &\quad + \sum_{i=1}^3 \{n(R'_i) + 1\}\pi^*(R'_i) + \sum_{P' \in S} n(P')\pi^*(P'), \end{aligned}$$

where  $S$  is the set of points  $P' \in E$  except  $P'_1$ ,  $P'_2$ ,  $Q'$  and  $R'_i$ 's. We set

$$\begin{aligned} D'_0 &= -P'_1 - P'_2 - Q' + \sum_{i=1}^3 R'_i, & D'_1 &= -2Q' + \sum_{i=1}^3 R'_i, \\ D'_2 &= -3Q' + P'_1 + \sum_{i=1}^3 R'_i, & D'_3 &= -4Q' + P'_1 + P'_2 + \sum_{i=1}^3 R'_i, \\ D'_4 &= -5Q' + 2P'_1 + P'_2 + \sum_{i=1}^3 R'_i, & D'_5 &= -6Q' + 2P'_1 + 2P'_2 + \sum_{i=1}^3 R'_i, \\ D'_6 &= -7Q' + 3P'_1 + 2P'_2 + \sum_{i=1}^3 R'_i. \end{aligned}$$

Then for  $r = 0, 1, \dots, 6$ ,  $f \in L(D'_r)$  implies that  $f dy / y^{1-r}$  is a holomorphic differential form, where

$$L(D'_r) = \{f \in K(E) \mid \operatorname{div}_E(f) \geq -D'_r\}.$$

Since we have

$$\sigma\left(\frac{dy}{y}\right) = \frac{d\sigma y}{\sigma y} = \frac{d\xi_7 y}{\xi_7 y} = \frac{dy}{y},$$

we regard  $dy/y$  as a differential form on  $E$ . Hence there exists  $f \in K(E)$  such that  $f dy/y$  is holomorphic. Then we must have

$$\operatorname{div}_E(f) = P'_1 + P'_2 + Q' - \sum_{i=1}^3 R'_i, \text{ i.e., } l(D'_0) = 1,$$

where for any divisor  $D$  we denote by  $l(D)$  the dimension of the  $k$ -vector space  $L(D)$ . Moreover, we have  $l(D'_r) = 1$  for  $r = 1, 2, \dots, 6$ , because of  $\deg(D'_r) = 1$  for  $r = 1, 2, \dots, 6$ . First we will show that  $l(D'_1 - P'_2) = 0$ . If  $l(D'_1 - P'_2) > 0$ , then we have

$$D'_1 - P'_2 \sim -2Q' + \sum_{i=1}^3 R'_i - P'_2 \sim 0 \sim D'_0 = -P'_1 - P'_2 - Q' + \sum_{i=1}^3 R'_i,$$

which implies that  $P'_1 \sim Q'$ . This is a contradiction. Now in view of  $2P'_1 \sim 2Q'$ , we have

$$D'_2 - P'_2 \sim -3Q' + P'_1 + \sum_{i=1}^3 R'_i - P'_2 \sim P'_1 - P'_2 - Q' + \sum_{i=1}^3 R'_i = D'_0,$$

which implies that  $l(D'_2 - P'_2) = 1$  and  $l(D'_2 - 2P'_2) = 0$ . If  $l(D'_3 - P'_2) > 0$ , then we have

$$-4Q' + P'_1 + \sum_{i=1}^3 R'_i \sim D'_3 - P'_2 \sim D'_0 = -P'_1 - P'_2 - Q' + \sum_{i=1}^3 R'_i$$

which implies that

$$P'_2 + 2Q' \sim P'_2 + 2P'_1 \sim 3Q', \text{ i.e., } P'_2 \sim Q'.$$

This is a contradiction. Hence we obtain  $l(D'_3 - P'_2) = 0$ . If  $l(D'_4 - P'_2) > 0$ , then we have  $4Q' \sim 3P'_1 + P'_2$ , which implies that  $4P'_1 \sim 3P'_1 + P'_2$ , i.e.,  $P'_1 \sim P'_2$ . This is a contradiction. Hence we obtain  $l(D'_4 - P'_2) = 0$ . If  $l(D'_5 - P'_2) > 0$ , then we have  $5Q' \sim 3P'_1 + 2P'_2$ , which implies that  $2Q' \sim P'_1 + P'_2$ . Hence we have  $6Q' \sim 3P'_1 + 3P'_2$ , which implies that  $P'_2 \sim Q'$ . This is a contradiction. Hence we obtain  $l(D'_5 - P'_2) = 0$ . If  $l(D'_6 - P'_2) > 0$ , then we have  $6Q' \sim 4P'_1 + 2P'_2$ , which implies that  $Q' \sim P'_2$ . This is a contradiction. Hence we obtain  $l(D'_6 - P'_2) = 0$ . For  $r = 0, 1, \dots, 6$  we take a non-zero element  $f_r \in L(D'_r)$  and we set  $\phi_r = f_r dy/y^{1-r}$ . Then by the above we see the following:

$$\begin{aligned} \operatorname{ord}_{P_2}(\phi_0) &= 6 = 7 - 1, & \operatorname{ord}_{P_2}(\phi_1) &= 2 = 3 - 1, & \operatorname{ord}_{P_2}(\phi_2) &= 12 = 13 - 1, \\ \operatorname{ord}_{P_2}(\phi_3) &= 1 = 2 - 1, & \operatorname{ord}_{P_2}(\phi_4) &= 4 = 5 - 1, & \operatorname{ord}_{P_2}(\phi_5) &= 0 = 1 - 1, \\ \operatorname{ord}_{P_2}(\phi_6) &= 3 = 4 - 1. \end{aligned}$$

Therefore  $N \setminus H(P_2)$  contains 7, 3, 13, 2, 5, 1 and 4. Hence the semigroup  $H(P_2)$  is generated by 6, 8, 9, 10 and 11. Q.E.D.

#### §4. On the numerical semigroup generated by 6, 7, 9 and 17

Let  $H$  be the numerical semigroup with  $M(H) = \{6, 7, 9, 17\}$ . We set

$$a_0 = 6, a_1 = 7, a_2 = 9, \text{ and } a_3 = 17.$$

Let  $\varphi_H$  be the  $k$ -algebra homomorphism from  $k[X] = k[X_0, X_1, X_2, X_3]$  to  $k[H] = k[t^h]_{h \in H}$  defined by sending  $X_i$  to  $t^{a_i}$  for each  $i$ . Moreover,  $I_H$  denotes the ideal  $\text{Ker } \varphi_H$ . Then we have the following :

**Proposition 4.1.** *The ideal  $I_H$  is generated by*

$$X_0X_3 - X_1^2X_2, X_0X_2^2 - X_1X_3, X_2^2 - X_0^3, X_3^2 - X_0^3X_1X_2, X_0^2X_1^2 - X_2X_3 \text{ and } X_1^3 - X_0^2X_2.$$

*Proof.* Let  $J$  be the ideal in  $k[X]$  generated by the above elements. We will show that  $I_H \subseteq J$ . We set

$$\alpha_i = \text{Min} \{ \alpha \in \mathbb{N} > 0 \mid \alpha a_i \in \langle a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_3 \rangle \}, \text{ all } i$$

where for non-negative integers  $b_1, \dots, b_n$ ,  $\langle b_1, \dots, b_n \rangle$  denotes the subsemigroup of  $\mathbb{N}$  generated by  $b_1, \dots, b_n$ . In view of

$$3a_0 = 2a_2, 3a_1 = 2a_0 + a_2 \text{ and } 2a_3 = 3a_0 + a_1 + a_2,$$

we have  $\alpha_0 = \alpha_1 = 3$  and  $\alpha_2 = \alpha_3 = 2$ . We may take as generators for the ideal  $I_H$  one of the following types :

$$(1) \quad F = X_i^{\nu_i} - X_j^{\mu_j} X_l^{\mu_l} X_m^{\mu_m}, \text{ where } i, j, l \text{ and } m \text{ are distinct,}$$

and  $\nu_i > 0, \mu_j > 0, \mu_l \geq 0, \mu_m \geq 0$ .

$$(2) \quad F = X_i^{\nu_i} X_j^{\nu_j} - X_l^{\mu_l} X_m^{\mu_m}, \text{ where } i, j, l \text{ and } m \text{ are distinct,}$$

and  $\nu_i > 0, \nu_j > 0, \mu_l > 0, \mu_m > 0$ .

We consider the case (1). If  $\nu_i > \alpha_i$ , then we may decrease the weighted degree of  $F$  or reduce this case to the case (2), because

$$X_0^{\alpha_0} - X_2^{\alpha_2}, X_1^{\alpha_1} - X_0^2X_2, X_3^{\alpha_3} - X_0^3X_1X_2 \in J,$$

where the weighted degree on  $k[X_0, X_1, X_2, X_3]$  is defined by the following : For any  $i$ , the weighted degree of  $X_i$  is  $\alpha_i$  and for any non-zero element  $c$  of  $k$  the weighted degree of  $c$  is zero. Hence we may assume that  $\nu_i = \alpha_i$ . Therefore, the generators of type (1) are contained in  $J$ .

Next we consider the case (2). Then we may assume that  $F$  is one of the following :

$$\text{a) } X_0^{\nu_0} X_1^{\nu_1} - X_2^{\mu_2} X_3^{\mu_3}, \quad \text{b) } X_0^{\nu_0} X_2^{\nu_2} - X_1^{\mu_1} X_3^{\mu_3}, \quad \text{c) } X_0^{\nu_0} X_3^{\nu_3} - X_1^{\mu_1} X_2^{\mu_2}.$$

In the case a) we may assume that  $\nu_0 < \alpha_0 = 3, \nu_1 < \alpha_1 = 3, \mu_2 < \alpha_2 = 2$  and  $\mu_3 < \alpha_3 = 2$ , otherwise we may decrease the weighted degree of  $F$ . Hence we get

$$26 = a_2 + a_3 = \nu_0 a_0 + \nu_1 a_1 = 6\nu_0 + 7\nu_1,$$

which implies that  $\nu_0=2$  and  $\nu_1=2$ . Hence we have  $F=X_0^2X_1^2-X_2X_3\in J$ . In the case b), similarly we may assume that  $\mu_1<3$  and  $\mu_3<2$ . Hence we have

$$\nu_0a_0+\nu_2a_2=24 \text{ or } 31,$$

which implies that  $\nu_0=1$  and  $\nu_2=2$ . Hence we have  $X_0X_2^2-X_1X_3\in J$ . In the case c) we may also assume that  $\nu_0<3$ ,  $\nu_3<2$ ,  $\mu_1<3$  and  $\mu_2<2$ . Hence we get  $6\nu_0+17=7\mu_1+9$ , which implies that  $\nu_0=1$  and  $\mu_1=2$ . Hence we have  $F=X_0X_3-X_1^2X_2\in J$ . Q.E.D.

Let  $S$  be the subsemigroup of  $\mathbf{Z}^6$  generated by  $b_1, b_2, \dots, b_{10}$ , where

$$\begin{aligned} b_1 &= (1, 0, \dots, 0), \quad b_2 = (0, 1, \dots, 0), \quad \dots, \quad b_6 = (0, 0, \dots, 1), \\ b_7 &= (1, 1, -1, -1, 0, 0), \quad b_8 = (0, 1, -1, 1, -1, 0), \quad b_9 = (-1, 1, 0, 1, -1, -1). \end{aligned}$$

Then we see the following :

**Lemma 4.2.** *The subsemigroup  $S$  of  $\mathbf{Z}^6$  is saturated, i.e., if  $nr\in S$  with  $n\in\mathbf{N}>0$  and  $r\in\mathbf{Z}^6$ , then  $r\in S$ .*

*Proof.* Take  $p=(p_1, \dots, p_6)\in\sum_{i=1}^9\mathbf{R}_+b_i\cap\mathbf{Z}^6$ , where  $\mathbf{R}_+$  denotes the set of non-negative real numbers. Then it suffices to show that  $p\in S$ . Moreover, we may assume that  $p=\sum_{i=1}^9m_ib_i$  with  $0\leq m_i<1$ , all  $i$ . Then we obtain

$$\begin{aligned} p_1 &= m_1+m_7-m_9\geq 0, \quad p_2 = m_2+m_7+m_8+m_9\geq 0, \quad p_3 = m_3-m_7-m_8\geq -1, \\ p_4 &= m_4-m_7+m_8+m_9\geq 0, \quad p_5 = m_5-m_8-m_9\geq -1, \quad p_6 = m_6-m_9=0. \end{aligned}$$

It suffices to show that  $p\in S$  if  $(p_3, p_5)=(-1, -1)$  or  $(-1, 0)$  or  $(0, -1)$ . Let  $(p_3, p_5)=(-1, -1)$ , i.e.,  $m_3+1=m_7+m_8$  and  $m_5+1=m_8+m_9$ . Hence we obtain

$$p_2 = m_2+m_3+1+m_9\geq 2 \text{ and } p_4 = m_4-m_7+m_5+1\geq 1.$$

Therefore we may assume that

$$p=(0, 2, -1, 1, -1, 0)=b_2+b_8\in S.$$

Let  $(p_3, p_5)=(-1, 0)$ , i.e.,  $m_3+1=m_7+m_8$  and  $m_5=m_8+m_9$ . Then

$$p_2\geq 1 \text{ and } p_1+p_4=m_1+m_4+m_8\geq 1,$$

which imply that  $p_1\geq 1$  or  $p_4\geq 1$ . Therefore we may assume that

$$p=(1, 1, -1, 0, 0, 0)=b_4+b_7\in S \text{ or } p=(0, 1, -1, 1, 0, 0)=b_5+b_8\in S.$$

Let  $(p_3, p_5)=(0, -1)$ , i.e.,  $m_3=m_7+m_8$  and  $m_5+1=m_8+m_9$ . Then we have

$$p_2 = m_2+m_7+m_5+1\geq 1 \text{ and } p_4 = m_4-m_7+m_5+1\geq 1.$$

Therefore we may assume that

$$p=(0, 1, 0, 1, -1, 0)=b_3+b_8\in S. \quad \text{Q.E.D.}$$

By Proposition 4.1 we have the following relations :

$$\begin{aligned} d_{01}a_0 + d_{31}a_3 &= d_{21}a_2 + (d_{11} + d_{12})a_1, \quad d_{02}a_0 + (d_{21} + d_{22})a_2 = d_{11}a_1 + d_{31}a_3, \\ (d_{21} + d_{22})a_2 &= (d_{01} + d_{03} + d_{04})a_0, \quad 2d_{31}a_3 = (d_{02} + d_{03} + d_{04})a_0 + d_{12}a_1 + d_{21}a_2, \\ (d_{03} + d_{04})a_0 &+ (d_{11} + d_{12})a_1 = d_{22}a_2 + d_{31}a_3, \quad (2d_{11} + d_{12})a_1 = (d_{01} + d_{02})a_0 + d_{22}a_2, \end{aligned}$$

where all  $d_{ij}$ 's are equal to 1. Hence we set

$$\begin{aligned} g_1 &= X_0^{d_{01}}, \quad g_2 = X_3^{d_{31}}, \quad g_3 = X_2^{d_{21}}, \quad g_4 = X_1^{d_{11}}, \quad g_5 = X_0^{d_{02}}, \\ g_6 &= X_0^{d_{03}}, \quad g_7 = X_1^{d_{12}}, \quad g_8 = X_2^{d_{22}} \quad \text{and} \quad g_9 = X_0^{d_{04}}. \end{aligned}$$

Let

$$\pi : k[Y] = k[Y_1, \dots, Y_9] \longrightarrow k[S] = k[T^s]_{s \in S} \quad (\text{resp. } \eta : k[Y] \longrightarrow k[X])$$

be the  $k$ -algebra homomorphism defined by  $\pi(Y_i) = T^{b_i}$  (resp.  $\eta(Y_i) = g_i$ ).

**Lemma 4.3.** *The ideal  $I_H$  is generated by the elements of the set  $\eta(\text{Ker } \pi)$ .*

*Proof.* Let

$$\zeta : k[N^6] = k[t_1, \dots, t_6] \longrightarrow k[H]$$

be the  $k$ -algebra homomorphism defined by  $\zeta(t_i) = t^{w(g_i)}$ , where  $w(g_i)$  denotes the weighted degree of  $g_i$  defined in the proof of Proposition 4.1. Then  $\zeta$  extends to  $\zeta' : k[S] \longrightarrow k[H]$ , because we have

$$\begin{aligned} w(g_1 g_2 g_3^{-1} g_4^{-1}) &= d_{01}a_0 + d_{31}a_3 - d_{21}a_2 - d_{11}a_1 = d_{12}a_1 = w(g_7), \\ w(g_2 g_3^{-1} g_4 g_5^{-1}) &= d_{31}a_3 - d_{21}a_2 + d_{11}a_1 - d_{02}a_0 = d_{22}a_2 = w(g_8), \\ w(g_1^{-1} g_2 g_4 g_5^{-1} g_6^{-1}) &= -d_{01}a_0 + d_{31}a_3 + d_{11}a_1 - d_{02}a_0 - d_{03}a_0 \\ &= (d_{21} + d_{22})a_2 - d_{01}a_0 - d_{03}a_0 = d_{04}a_0 = w(g_9). \end{aligned}$$

Then we have  $\varphi_H \circ \eta = \zeta' \circ \pi$ , which implies that  $\eta(\text{Ker } \pi) \subseteq \text{Ker } \varphi_H = I_H$ . Hence it suffices to show that the generators for  $I_H$  as in Proposition 4.1 are contained in the ideal  $(\eta(\text{Ker } \pi))$  generated by the elements of  $\eta(\text{Ker } \pi)$ . Now we have the following :

$$\begin{aligned} \pi(Y_1 Y_2 - Y_4 Y_7 Y_3) &= T^{b_1+b_2} - T^{b_4+b_7+b_3} = 0 \quad \text{and} \\ \eta(Y_1 Y_2 - Y_4 Y_7 Y_3) &= g_1 g_2 - g_4 g_7 g_3 = X_0 X_3 - X_1^2 X_2, \\ \pi(Y_5 Y_3 Y_8 - Y_4 Y_2) &= T^{b_5+b_3+b_8} - T^{b_4+b_2} = 0 \quad \text{and} \\ \eta(Y_5 Y_3 Y_8 - Y_4 Y_2) &= g_5 g_3 g_8 - g_4 g_2 = X_0 X_2^2 - X_1 X_3, \\ \pi(Y_3 Y_8 - Y_6 Y_1 Y_9) &= T^{b_3+b_8} - T^{b_6+b_1+b_9} = 0 \quad \text{and} \\ \eta(Y_3 Y_8 - Y_6 Y_1 Y_9) &= g_3 g_8 - g_6 g_1 g_9 = X_2^2 - X_0^3, \\ \pi(Y_2^2 - X_6 Y_5 Y_9 Y_7 Y_3) &= T^{2b_2} - T^{b_6+b_5+b_9+b_7+b_3} = 0 \quad \text{and} \\ \eta(Y_2^2 - Y_6 Y_5 Y_9 Y_7 Y_3) &= g_2^2 - g_6 g_5 g_9 g_7 g_3 = X_3^2 - X_0^3 X_1 X_2, \\ \pi(Y_6 Y_9 Y_4 Y_7 - Y_8 Y_2) &= T^{b_6+b_9+b_4+b_7} - T^{b_8+b_2} = 0 \quad \text{and} \\ \eta(Y_6 Y_9 Y_4 Y_7 - Y_8 Y_2) &= g_6 g_9 g_4 g_7 - g_8 g_2 = X_0^2 X_1^2 - X_2 X_3, \\ \pi(Y_4^2 Y_7 - Y_1 Y_5 Y_8) &= T^{2b_4+b_7} - T^{b_1+b_5+b_8} = 0 \quad \text{and} \\ \eta(Y_4^2 Y_7 - Y_1 Y_5 Y_8) &= g_4^2 g_7 - g_1 g_5 g_8 = X_1^3 - X_0^2 X_2. \end{aligned}$$

Hence we obtain  $\eta(\text{Ker } \pi) = I_H$ .

*Q.E.D.*



Hence we see the following :

**Theorem 4.4.** *The numerical semigroup  $H$  with  $M(H) = \{6, 7, 9, 17\}$  is Weierstrass.*

*Proof.* By Lemmas 4.2 and 4.3,  $H$  is Weierstrass<sup>2)</sup>.

*Q.E.D.*

### §5. On the numerical semigroup generated by 7, 8, 9, 10 and 11

Let  $H$  be the numerical semigroup with  $M(H) = \{7, 8, 9, 10, 11\}$ . We set

$$a_0 = 7, a_1 = 8, a_2 = 9, a_3 = 10 \text{ and } a_4 = 11.$$

Let  $\varphi_H$  be the  $k$ -algebra homomorphism from  $k[X] = k[X_0, X_1, X_2, X_3, X_4]$  to  $k[H] = k[t^h]_{h \in H}$  defined by sending  $X_i$  to  $t^{a_i}$  for each  $i$ . Moreover,  $I_H$  denotes the ideal  $\text{Ker } \varphi_H$ . Then we have the following :

**Proposition 5.1.** *The ideal  $I_H$  is generated by*

$$X_0^3 - X_3X_4, X_1^2 - X_0X_2, X_2^2 - X_1X_3, X_3^2 - X_2X_4, X_4^2 - X_0^2X_1, X_0X_3 - X_1X_2, X_0X_4 - X_1X_3 \text{ and } X_1X_4 - X_2X_3.$$

*Proof.* We have

$$a_1 = a_0 + 1, a_2 = a_1 + 1, a_3 = a_2 + 1 \text{ and } a_4 = a_3 + 1.$$

Hence the ideal  $I_H$  contains

$$X_0X_2 - X_1^2, X_0X_3 - X_1X_2, X_0X_4 - X_1X_3, X_1X_3 - X_2^2, \\ X_1X_4 - X_2X_3, X_2X_4 - X_3^2, X_3X_4 - X_0^3 \text{ and } X_4^2 - X_0^2X_1.$$

Let  $J$  be the ideal generated by the above elements. We will show that  $I_H$  is contained in  $J$ . We may take as generators for the ideal  $I_H$  the following type :

$$F = \prod_i X_i^{\nu_i} - \prod_i X_i^{\mu_i}, \quad \nu_i, \mu_i = 0 \text{ for all } i.$$

We set  $L = \prod_i X_i^{\nu_i}$  and  $R = \prod_i X_i^{\mu_i}$ . Then we may assume that  $F$  is one of the following types :

- (1)  $L = X_0L_0$  and  $R = X_iX_jR_0$ ,  $1 \leq i \leq j$ , where  $R_0$  does not contain  $X_0$ .
- (2) Neither  $L$  nor  $R$  contains  $X_0$ .

We consider the case (1). If  $(i, j) = (1, 1), (1, 2), (1, 3), (2, 2), (3, 4)$  or  $(4, 4)$ , then we have  $X_iX_j - X_0X_l \in J$ . Hence we may decrease the weighted degree of  $F$ , where the weighted degree on  $k[X_0, X_1, X_2, X_3, X_4]$  is defined by the following : For any  $i$ , the weighted degree of  $X_i$  is  $a_i$  and for any non-zero element  $c$  of  $k$  the weighted degree of  $c$  is zero. Since  $J$  contains  $X_1X_4 - X_2X_3$  and  $X_2X_4 - X_3^2$ , we may assume that  $(i, j) = (1, 4)$  or  $(2, 4)$ . Then  $R_0$  is a non-constant monomial, because of  $19 - 7 = 12 \notin H$  and  $20 - 7 \notin H$ . Hence we set  $R_0 = X_lR_1$ ,  $l \geq 1$ . Then we have

$$X_i X_j R_0 = X_i X_j X_l R_i \equiv X_0 R_2 \pmod{J},$$

which implies that we may decrease the weighted degree of  $F$ . Therefore in the case (1) we have  $F \in J$ .

Next we consider the case (2). If  $(i, j)$  (resp.  $(l, m)$ ) =  $(1, 1)$ ,  $(1, 2)$ ,  $(1, 3)$ ,  $(2, 2)$ ,  $(3, 4)$  or  $(4, 4)$ , then this case is reduced to the case (1). Hence we may assume that  $(i, j)$  (resp.  $(l, m)$ ) is one of the following:  $(1, 4)$ ,  $(2, 3)$ ,  $(2, 4)$ ,  $(3, 3)$ . In the above cases we may decrease the weighted degree of  $F$ . Therefore in the case (2) we have  $F \in J$ . Hence we obtain  $I_H = J$ . Q.E.D.

Let  $S$  be the subsemigroup of  $\mathbf{Z}^6$  generated by  $b_1, b_2, \dots, b_{10}$ , where

$$\begin{aligned} b_1 &= (1, 0, \dots, 0), \quad b_2 = (0, 1, \dots, 0), \quad \dots, \quad b_6 = (0, 0, \dots, 1), \quad b_7 = (1, 1, -1, 0, 0, 0), \\ b_8 &= (-1, 0, 0, 1, 1, 0), \quad b_9 = (-1, 0, 0, 0, 1, 1) \text{ and } b_{10} = (-1, 0, 1, 0, 1, 0). \end{aligned}$$

Then we see the following:

**Lemma 5.2.** *The subsemigroup  $S$  of  $\mathbf{Z}^6$  is saturated.*

*Proof.* Take  $p = (p_1, \dots, p_6) \in \sum_{i=1}^{10} \mathbf{R}_+ b_i \cap \mathbf{Z}^6$ . Then it suffices to show that  $p \in S$ . Moreover, we may assume that  $p = \sum_{i=1}^{10} m_i b_i$  with  $0 \leq m_i < 1$ , all  $i$ . Then we obtain

$$\begin{aligned} p_1 &= m_1 + m_7 - m_8 - m_9 - m_{10} \geq -2, \quad p_2 = m_2 + m_7 \geq 0, \quad p_3 = m_3 - m_7 + m_{10} \geq 0, \\ p_4 &= m_4 + m_8 \geq 0, \quad p_5 = m_5 + m_8 + m_9 + m_{10} \geq 0, \quad p_6 = m_6 + m_9 \geq 0. \end{aligned}$$

It suffices to show that if  $p_1 = -2$  or  $-1$ , then  $p \in S$ . Let  $p_1 = -2$ , i.e.,  $m_1 + m_7 + 2 = m_8 + m_9 + m_{10}$ . Hence we have

$$\begin{aligned} p_3 &= m_3 - m_7 - m_8 - m_9 + m_1 + m_7 + 2 = m_1 + m_3 - m_8 - m_9 + 2 \geq 1 \\ p_4 &\geq 1, \quad p_5 = m_5 + m_1 + m_7 + 2 \geq 2, \quad p_6 \geq 1. \end{aligned}$$

Hence we may assume that

$$p = (-2, 0, 1, 1, 2, 1) = b_4 + b_9 + b_{10} \in S.$$

Let  $p_1 = -1$ , i.e.,  $m_1 + m_7 + 1 = m_8 + m_9 + m_{10}$ . Hence we have  $p_5 = m_5 + m_1 + m_7 + 1 \geq 1$  and  $p_4 + p_6 = m_4 + m_6 + m_1 + m_7 + 1 - m_{10} \geq 1$ , which imply that  $p_4 \geq 1$  or  $p_6 \geq 1$ . If  $p_4 \geq 1$ , we may assume that

$$p = (-1, 0, 0, 1, 1, 0) = b_8 \in S.$$

If  $p_6 \geq 1$ , we may assume that

$$p = (-1, 0, 0, 0, 1, 1) = b_9 \in S. \quad \text{Q.E.D.}$$

By Proposition 5.1, we have the following relations:

$$\begin{aligned}
(c_{10} + c_{40})a_0 &= c_{03}a_3 + c_{04}a_4, & (c_{21} + c_{41})a_1 &= c_{10}a_0 + c_{12}a_2, \\
(c_{12} + c_{32})a_2 &= c_{21}a_1 + c_{23}a_3, & (c_{03} + c_{23})a_3 &= c_{32}a_2 + c_{34}a_4, \\
(c_{04} + c_{34})a_4 &= c_{40}a_0 + c_{41}a_1, & c_{10}a_0 + c_{23}a_3 &= c_{41}a_1 + c_{32}a_2, \\
c_{10}a_0 + c_{34}a_4 &= c_{41}a_1 + c_{03}a_3, & c_{21}a_1 + c_{34}a_4 &= c_{12}a_2 + c_{03}a_3,
\end{aligned}$$

where we set

$$c_{10}=1, c_{40}=2, c_{03}=c_{21}=c_{41}=c_{32}=c_{04}=c_{12}=c_{23}=c_{34}=1.$$

Hence we set

$$\begin{aligned}
g_1 &= X_0^{c_{10}}, & g_2 &= X_0^{c_{40}}, & g_3 &= X_3^{c_{03}}, & g_4 &= X_1^{c_{21}}, & g_5 &= X_1^{c_{41}}, \\
g_6 &= X_2^{c_{32}}, & g_7 &= X_4^{c_{04}}, & g_8 &= X_2^{c_{12}}, & g_9 &= X_3^{c_{23}}, & g_{10} &= X_4^{c_{34}}.
\end{aligned}$$

Let

$$\pi : k[Y] = k[Y_1, \dots, Y_{10}] \longrightarrow k[S] = k[T^s]_{s \in S} \quad (\text{resp. } \eta : k[Y] \longrightarrow k[X])$$

be the  $k$ -algebra homomorphism defined by  $\pi(Y_i) = T^{b_i}$  (resp.  $\eta(Y_i) = g_i$ ).

**Lemma 5.3.** *The ideal  $I_H$  is generated by the elements of the set  $\eta(\text{Ker } \pi)$ .*

*Proof.* Let

$$\zeta : k[N^6] = k[t_1, \dots, t_6] \longrightarrow k[H]$$

be the  $k$ -algebra homomorphism defined by  $\zeta(t_i) = t^{w(g_i)}$ , where  $w(g_i)$  denotes the weighted degree of  $g_i$  defined in the proof of Proposition 5.1. Then  $\zeta$  extends to  $\zeta' : k[S] \longrightarrow k[H]$ , because we have

$$\begin{aligned}
w(g_1 g_2 g_3^{-1}) &= c_{10}a_0 + c_{40}a_0 - c_{03}a_3 = c_{04}a_4 = w(g_7), \\
w(g_1^{-1} g_4 g_5) &= -c_{10}a_0 + c_{21}a_1 + c_{41}a_1 = c_{12}a_2 = w(g_8), \\
w(g_1^{-1} g_5 g_6) &= -c_{10}a_0 + c_{41}a_1 + c_{32}a_2 = c_{23}a_3 = w(g_9), \\
w(g_1^{-1} g_3 g_5) &= -c_{10}a_0 + c_{03}a_3 + c_{41}a_1 = c_{34}a_4 = w(g_{10}).
\end{aligned}$$

Then we have  $\varphi_H \circ \eta = \zeta' \circ \pi$ , which implies that  $\eta(\text{Ker } \pi) \subseteq \text{Ker } \varphi_H = I_H$ . Hence it suffices to show that the generators for  $I_H$  as in Proposition 5.1 are contained in the ideal  $(\eta(\text{Ker } \pi))$  generated by the elements of  $\eta(\text{Ker } \pi)$ . Now we have the following:

$$\begin{aligned}
\pi(Y_1 Y_2 - Y_3 Y_7) &= T^{b_1+b_2} - T^{b_3+b_7} = 0 \text{ and} \\
\eta(Y_1 Y_2 - Y_3 Y_7) &= g_1 g_2 - g_3 g_7 = X_0^3 - X_3 X_4. \\
\pi(Y_4 Y_5 - Y_1 Y_8) &= T^{b_4+b_5} - T^{b_1+b_8} = 0 \text{ and} \\
\eta(Y_4 Y_5 - Y_1 Y_8) &= g_4 g_5 - g_1 g_8 = X_1^2 - X_0 X_2. \\
\pi(Y_8 Y_6 - Y_4 Y_9) &= T^{b_8+b_6} - T^{b_4+b_9} = 0 \text{ and} \\
\eta(Y_8 Y_6 - Y_4 Y_9) &= g_8 g_6 - g_4 g_9 = X_2^2 - X_1 X_3. \\
\pi(Y_3 Y_9 - Y_6 Y_{10}) &= T^{b_3+b_9} - T^{b_6+b_{10}} = 0 \text{ and} \\
\eta(Y_3 Y_9 - Y_6 Y_{10}) &= g_3 g_9 - g_6 g_{10} = X_3^2 - X_2 X_4. \\
\pi(Y_7 Y_{10} - Y_2 Y_5) &= T^{b_7+b_{10}} - T^{b_2+b_5} = 0 \text{ and} \\
\eta(Y_7 Y_{10} - Y_2 Y_5) &= g_7 g_{10} - g_2 g_5 = X_4^2 - X_0^2 X_1. \\
\pi(Y_1 Y_9 - Y_5 Y_6) &= T^{b_1+b_9} - T^{b_5+b_6} = 0 \text{ and} \\
\eta(Y_1 Y_9 - Y_5 Y_6) &= g_1 g_9 - g_5 g_6 = X_0 X_3 - X_1 X_2. \\
\pi(Y_1 Y_{10} - Y_5 Y_3) &= T^{b_1+b_{10}} - T^{b_5+b_3} = 0 \text{ and} \\
\eta(Y_1 Y_{10} - Y_5 Y_3) &= g_1 g_{10} - g_5 g_3 = X_0 X_4 - X_1 X_3. \\
\pi(Y_4 Y_{10} - Y_8 Y_3) &= T^{b_4+b_{10}} - T^{b_8+b_3} = 0 \text{ and} \\
\eta(Y_4 Y_{10} - Y_8 Y_3) &= g_4 g_{10} - g_8 g_3 = X_1 X_4 - X_2 X_3.
\end{aligned}$$

Hence we obtain  $(\eta(\text{Ker } \pi)) = I_H$ .

*Q.E.D.*

Hence we see the following:

**Theorem 5.4.** *The numerical semigroup  $H$  with  $M(H) = \{7, 8, 9, 10, 11\}$  is Weierstrass.*

*Proof.* By Lemmas 5.2 and 5.3,  $H$  is Weierstrass<sup>2)</sup>.

*Q.E.D.*

### References

- 1) C. Maclachlan: Weierstrass points on compact Riemann surfaces. J. London Math. Soc. **3** (1971) 722-724.
- 2) J. Komeda: On Weierstrass points whose first non-gaps are four. J. reine angew. Math **341** (1983) 68-86.
- 3) J. Komeda: On the existence of Weierstrass points whose first non-gaps are five. In preparation.
- 4) D. Eisenbud and J. Harris: Existence, decomposition, and limits of certain Weierstrass points. Invent. Math. **87** (1987) 495-515.
- 5) J. Komeda: On primitive Schubert indices of genus  $g$  and weight  $g-1$ . J. Math. Soc. Japan **43** (1991) 437-445.
- 6) J. Komeda: On primitive Schubert indices. Research Reports of Kanagawa Institute of Technology **B-14** (1990) 245-251.