

On the Weierstrass points whose semigroups are generated by two elements

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Abstract

Let $M_{g,1}$ be the moduli space of pointed curves of genus g where a *curve* means a complete non-singular 1-dimensional algebraic variety over an algebraically closed field of characteristic 0. For any $(C, P) \in M_{g,1}$, $H(P)$ denotes the set of integers which are pole orders at P of regular functions on $C \setminus \{P\}$. We set $C_H = \{(C, P) \in M_{g,1} \mid H(P) = H\}$. In this paper we calculate the dimension of C_H for any semigroup H generated by two relatively prime positive integers.

§1. Preliminaries.

Let X be an affine scheme over a field k with only isolated singularities. Let \mathbf{C} be the category of Artinian local k -algebras A with $A/\mathfrak{m}_A = k$ where \mathfrak{m}_A is the maximal ideal in A . By an (infinitesimal) *deformation* of X/k to A we mean a fiber product diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ \text{Spec } k & \rightarrow & \text{Spec } A \end{array} \quad X \xrightarrow{\sim} Y \times_{\text{Spec } A} \text{Spec } k$$

where Y is flat over $\text{Spec } A$, which is abbreviated to the morphism $Y \rightarrow \text{Spec } A$. Define the deformation functor $\mathbf{D}_X: \mathbf{C} \rightarrow \text{Sets}$ by setting

$$\mathbf{D}_X(A) = \text{Set of isomorphism classes of deformations of } X/k \text{ to } A.$$

Since X has only isolated singularities, \mathbf{D}_X has a *hull*⁽¹⁾, that is to say, there exist a complete noetherian local k -algebra R for which $R/\mathfrak{m}_R^n \in \mathbf{C}$, all $n \geq 1$, and $\xi \in \hat{\mathbf{D}}_X(R) = \text{proj } \text{Lim } \mathbf{D}_X(R/\mathfrak{m}_R^n)$ such that

(1) for any surjective morphism $p: A \rightarrow B$ in \mathbf{C} , the map

$$h_R(A) \rightarrow h_R(B) \times \mathbf{D}_X(A)$$

defined by sending u to $(u \circ p, \varphi(A)(u))$ is a surjection where we set

$h_R(A) = \text{Hom local } k \text{ alg. } (R, A)$ and φ is the morphism of functors induced by ξ , and

(2) the induced map $\varphi(k[\epsilon]): h_R(k[\epsilon]) \rightarrow \mathbf{D}_X(k[\epsilon])$ is a bijection.

In this case, R is called the *formal moduli space* of X , or the *parameter space* of X .

Let H be a *numerical semigroup*, i.e., a subsemigroup of the additive semigroup \mathbf{N} of the non-negative integers whose complement $\mathbf{N} \setminus H$ in \mathbf{N} is finite. We set $X = \text{Spec } k[H]$. Let

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$M(H) = \{a_0, a_1, \dots, a_n\}$ be the minimal set of generators for H . Let

$$\varphi_H: P = k[X_0, X_1, \dots, X_n] \longrightarrow k[H] = k[t^h]_{h \in H}$$

be the k -algebra homomorphism defined by $\varphi_H(X_i) = t^{a_i}$. We set $I = I_H = \text{Ker } \varphi_H = (f_1, \dots, f_N)$ and $B = P/I$. D denotes the B -submodule of $\text{Hom}_{B\text{-mod.}}(I/I^2, B)$ generated by the homomorphisms $d_{(0)}, d_{(1)}, \dots, d_{(n)}$, where $d_{(l)}$ is defined by sending $h + I^2$ to $\partial h / \partial X_l + I$, all l . Then we have the exact sequence of k -vector spaces

$$0 \longrightarrow D \longrightarrow \text{Hom}_{B\text{-mod.}}(I/I^2, B) \longrightarrow \mathbf{D}_X(k[\epsilon]) \longrightarrow 0,$$

where the above morphism $\text{Hom}_{B\text{-mod.}}(I/I^2, B) \longrightarrow \mathbf{D}_X(k[\epsilon])$, which is denoted by Φ , is defined by sending θ with $\theta(f_i + I^2) = g_i + I$ to $cI(\text{Spec } k[\epsilon][X_0, \dots, X_n] / (f_1 + \epsilon g_1, \dots, f_N + \epsilon g_N)) \longrightarrow \text{Spec } k[\epsilon]$.

We define the weighted degree w on $P = k[X_0, \dots, X_n]$ as follows: $w(X_i) = a_i$ and for all $c \in k \setminus \{0\}$, $w(c) = 0$. Let d_i be the weighted degree of f_i , $i = 1, \dots, N$. For all $\nu \in \mathbf{Z}$, $T_X^1(\nu)$ denotes

$$\Phi(\{\theta \in \text{Hom}_{B\text{-mod.}}(I/I^2, B) \mid \text{If } \theta(f_i + I^2) = g_i + I, \text{ then } g_i + I \text{ is a homogeneous element of weighted degree } d_i + \nu\}).$$

Then we have $\mathbf{D}_X(k[\epsilon]) = \bigoplus_{\nu \in \mathbf{Z}} T_X^1(\nu)$. Since X has a hull, $\mathbf{D}_X(k[\epsilon])$ is a finite dimensional k -vector space. Hence we set $r = \dim \mathbf{D}_X(k[\epsilon])$. Choose a homogeneous k -basis $\{v_j\}_{1 \leq j \leq r}$ of $\mathbf{D}_X(k[\epsilon])$, $v_j \in T_X^1(e_j)$. We set $S = k[[t_1, \dots, t_r]]$ with its maximal ideal \mathbf{m} . We define the weighted degree w on S as follows: $w(t_i) = -e_i$ and for all $c \in k \setminus \{0\}$, $w(c) = 0$. Recall that to get the formal moduli space R of X , Schlessinger¹⁾ constructs by induction on q an ideal $J_q \subset S$, $J_2 = \mathbf{m}^2$, and a deformation $\xi_q \in \mathbf{D}_X(S/J_q)$ such that J_q is the minimal ideal J' satisfying the following two conditions:

- (1) $\mathbf{m}J_{q-1} \subset J' \subset J_{q-1}$.
- (2) There exists a deformation $\xi \in \mathbf{D}_X(S/J')$ inducing ξ_{q-1} .

Then pick any $\xi_q \in \mathbf{D}_X(S/J')$ inducing ξ_{q-1} . Finally set $J = \bigcap_{q=2}^\infty J_q$ and $R = S/J$. Let R' be the quotient of R obtained by setting all t_j with $e_j > 0$ equal to zero. Then $\mathbf{G}_m = \text{Spec } k[T, T^{-1}]$ acts on $\text{Spec } R'$ by sending the element $t_i \in R'$ to the element $T^{-e_i} \otimes t_i \in k[T, T^{-1}] \otimes R'$. Then Pinkham²⁾ showed the following:

If k is an algebraically closed field of characteristic 0, then there exists a \mathbf{G}_m -invariant open subset U of $\text{Spec } R'$ such that C_H is isomorphic to U/\mathbf{G}_m .

§2. The irreducibility of the moduli C_H where H is generated by two elements.

In this section we show that the moduli C_H is irreducible for any numerical semigroup H generated by two elements. The following plays an important role in the proof of the irreducibility of C_H and the calculation of its dimension in our cases.

Proposition. *Let k be a field. Take an element f of $P = k[X_0, X_1, \dots, X_n]$. Let $B = P/(f)$ and $X = \text{Spec } B$. If $P/(f, \partial f / \partial X_0, \partial f / \partial X_1, \dots, \partial f / \partial X_n)$ is an Artinian ring, then the parameter space of X is $k[[T_1, \dots, T_r]]$ where*

$$r = \dim_k \mathbf{D}_X(k[\epsilon]) = \dim_k P / (f, \partial f / \partial X_0, \partial f / \partial X_1, \dots, \partial f / \partial X_n).$$

Proof. Let $\tilde{I} = (f, \partial f / \partial X_0, \partial f / \partial X_1, \dots, \partial f / \partial X_n)$. Then we have an isomorphism $\pi: P / \tilde{I} \longrightarrow \mathbf{D}_X(k[\epsilon])$ of k -vector spaces defined by sending $g + \tilde{I}$ to $cl(\text{Spec } P' / I' \longrightarrow \text{Spec } k[\epsilon])$ where $P' = k[\epsilon][X_0, X_1, \dots, X_n]$ and $I' = (f + \epsilon g)$. Let $\{g_1 + \tilde{I}, \dots, g_r + \tilde{I}\}$ be a k -basis for P / \tilde{I} . We set

$$v_j = cl(\text{Spec } P' / I'_j \longrightarrow \text{Spec } k[\epsilon]) = \pi(g_j + \tilde{I}) \text{ for } j = 1, \dots, r,$$

where we set $I'_j = (f + \epsilon g_j)$. Then v_1, \dots, v_r form a k -basis for $\mathbf{D}_X(k[\epsilon])$. Let $S = k[[T_1, \dots, T_r]]$ with its maximal ideal \mathbf{m} . By the assumption X has only isolated singularities, which implies that \mathbf{D}_X has a hull. Hence we have canonical bijections

$$\overbrace{\mathbf{D}_X(k[\epsilon]) \times \dots \times \mathbf{D}_X(k[\epsilon])}^{r \text{ times}} \xrightarrow{\sim} \mathbf{D}_X(k[\epsilon] \times_k \dots \times_k k[\epsilon]) \xrightarrow{\sim} \mathbf{D}_X(S / \mathbf{m}^2).$$

Let $J_2 = \mathbf{m}^2$ and $\xi_2 \in \mathbf{D}_X(S / \mathbf{m}^2)$ be the image of (v_1, \dots, v_r) for the above composite map. Then we have

$$\xi_2 = cl(\text{Spec } P'' / I'' \longrightarrow \text{Spec } S / \mathbf{m}^2),$$

where $P'' = S / \mathbf{m}^2[X_0, X_1, \dots, X_n]$ and $I'' = (f + \sum_{j=1}^r (T_j + \mathbf{m}^2)g_j)$.

Let J_q and ξ_q , $q \geq 3$ be as in §1. We shall show that $J_q = \mathbf{m}^q$ for all $q \geq 2$ by induction on q . By induction hypothesis let $J_q = \mathbf{m}^q$ and

$$\xi_q = cl(\text{Spec } P^{(q)} / I^{(q)} \longrightarrow \text{Spec } S / \mathbf{m}^q) \in \mathbf{D}_X(S / \mathbf{m}^q),$$

where $P^{(q)} = (S / \mathbf{m}^q)[X_0, X_1, \dots, X_n]$ and $I^{(q)}$ is the ideal in $P^{(q)}$ generated by the element $f + \sum_{j=1}^r (T_j + \mathbf{m}^q)g_j$. Then we have fiber product diagrams

$$\begin{array}{ccccc} \text{Spec } B & \longrightarrow & \text{Spec } P'' / I'' & \longrightarrow & \text{Spec } P^{(q)} / I^{(q)}, \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & \text{Spec } S / \mathbf{m}^2 & \longrightarrow & \text{Spec } S / \mathbf{m}^q. \end{array}$$

Hence we have fiber product diagrams

$$\begin{array}{ccccc} \text{Spec } B & \longrightarrow & \text{Spec } P^{(q)} / I^{(q)} & \longrightarrow & \text{Spec } P^{(q+1)} / I^{(q+1)}, \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & \text{Spec } S / \mathbf{m}^q & \longrightarrow & \text{Spec } S / \mathbf{m}^{q+1}. \end{array}$$

To prove that $J_{q+1} = \mathbf{m}^{q+1}$ it suffices to show that $P^{(q+1)} / I^{(q+1)}$ is flat over S / \mathbf{m}^{q+1} . We have an exact sequence of $P^{(q)}$ -modules

$$0 \longrightarrow P^{(q)} \xrightarrow{\alpha^{(q)}} P^{(q)} \longrightarrow P^{(q)} / I^{(q)} \longrightarrow 0,$$

where $\alpha^{(q)}$ is defined by sending 1 to $f + \sum_{j=1}^r (T_j + \mathbf{m}^q)g_j$. Then $\alpha^{(q)}$ can be lifted, i.e., there exists an exact sequence of $P^{(q+1)}$ -modules

$$0 \longrightarrow P^{(q+1)} \xrightarrow{\alpha^{(q+1)}} P^{(q+1)} \longrightarrow P^{(q+1)}/I^{(q+1)} \longrightarrow 0$$

such that we have commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & P^{(q+1)} & \xrightarrow{\alpha^{(q+1)}} & P^{(q+1)} & \longrightarrow & P^{(q+1)}/I^{(q+1)} \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P^{(q)} & \xrightarrow{\alpha^{(q)}} & P^{(q)} & \longrightarrow & P^{(q)}/I^{(q)} \longrightarrow 0. \end{array}$$

In fact, let $\alpha^{(q+1)}: P^{(q+1)} \longrightarrow P^{(q+1)}$ be the homomorphism defined by $\alpha^{(q+1)}(1) = f + \sum_{j=1}^r (T_j + \mathfrak{m}^{q+1})g_j$. Then we obtain the desired commutative diagrams. Hence $P^{(q+1)}/I^{(q+1)}$ is flat over S/\mathfrak{m}^{q+1} , because by induction hypothesis $P^{(q)}/I^{(q)}$ is flat over S/\mathfrak{m}^q . Therefore the canonical map $\mathbf{D}_X(S/\mathfrak{m}^{q+1}) \longrightarrow \mathbf{D}_X(S/\mathfrak{m}^q)$ send $cl(\text{Spec } P^{(q+1)}/I^{(q+1)}) \longrightarrow \text{Spec } S/\mathfrak{m}^{q+1}$ to ξ_q , which implies that $J_{q+1} = \mathfrak{m}^{q+1}$.

By §1 the parameter space of X is $R = S/J$ with $J = \bigcap_{q=2}^{\infty} \mathfrak{m}^q$. Since $S = k[[T_1, \dots, T_r]]$ is a noetherian domain, we obtain $J = (0)$, which implies that $R = k[[T_1, \dots, T_r]]$. *Q.E.D.*

Theorem 1. *Let k be an algebraically closed field of characteristic 0. If H is a numerical semigroup generated by two elements, then the moduli C_H is irreducible.*

Proof. Let $M(H) = \{a_0, a_1\}$ and $\varphi_H: k[X_0, X_1] \longrightarrow k[t]$ be the k -algebra homomorphism defined by $\varphi_H(X_i) = t^{a_i}$, $i=0, 1$. Then we have $\text{Ker } \varphi_H = (f)$ with $f = X_0^{a_1} - X_1^{a_0}$. Hence by Proposition the parameter space of $X = \text{Spec } B$ with $B = k[X_0, X_1]/(f)$ is $k[[T_1, \dots, T_r]]$ where

$$r = \dim_k \mathbf{D}_X(k[\epsilon]) = \dim_k k[X_0, X_1] / (X_0^{a_1} - X_1^{a_0}, a_1 X_0^{a_1-1}, a_0 X_1^{a_0-1}).$$

By §1 there exists a \mathbf{G}_m -invariant open subsect U of $\text{Spec } k[T_{i_1}, T_{i_2}, \dots, T_{i_s}]$ for some subset $\{i_1, i_2, \dots, i_s\}$ of $\{1, 2, \dots, r\}$ such that C_H is isomorphic to U/\mathbf{G}_m . Since H is generated by two elements, we obtain $C_H \neq \emptyset^{(2)}$, which implies that C_H is irreducible. *Q.E.D.*

§3. The dimension of the moduli C_H where H is generated by two elements.

In the last section we calculate the dimension of the moduli C_H for any numerical semigroup H generated by two elements where k is an algebraically closed field of characteristic 0.

Lemma. *Let H be a numerical semigroup generated by a_0, a_1 with $a_0 < a_1$. Let $X = \text{Spec } B$ with $B = k[X_0, X_1]/I$ where I is the kernel of the k -algebra homomorphism $\varphi_H: k[X_0, X_1] \longrightarrow k[t]$ defined by $\varphi_H(X_i) = t^{a_i}$, $i=0, 1$. Then we have*

$$\mathbf{D}_X(k[\epsilon]) = \bigoplus_{0 \leq \nu_0 \leq a_1-2} \bigoplus_{0 \leq \nu_1 \leq a_0-2} T_X^1(\nu_0 a_0 + \nu_1 a_1 - a_0 a_1)$$

and

$$T_X^1(\nu_0 a_0 + \nu_1 a_1 - a_0 a_1) = k\Phi(\theta_{\nu_0 \nu_1}) \neq 0$$

for $0 \leq \nu_0 \leq a_1-2$ and $0 \leq \nu_1 \leq a_0-2$, where

$$\Phi : \text{Hom}_{B\text{-mod.}}(I/I^2, B) \longrightarrow \mathbf{D}_X(k[\epsilon])$$

is the homomorphism of k -vector spaces defined in §1 and $\theta_{\nu_0\nu_1} : I/I^2 \longrightarrow B$ is defined by sending $f+I^2$ to $X_0^{\nu_0}X_1^{\nu_1}+I$.

Proof. By §1, we have

$$T_X^1(\nu) = \Phi(\{\theta \in \text{Hom}_{B\text{-mod.}}(I/I^2, B) \mid \text{If } \theta(f+I^2) = h+I, \text{ then } h+I \text{ is a homogeneous element of weighted degree } a_0a_1+\nu\}).$$

Let $d_{(i)} : I/I^2 \longrightarrow B$ be the B -module homomorphism defined by sending $f+I^2$ to $\partial f/\partial X_i + I$ for $i=0, 1$. For any $\nu_0 \geq a_1-1$ or any $\nu_1 \geq a_0-1$, we have $\theta_{\nu_0\nu_1} \in \text{Ker } \Phi$, because of $\text{Ker } \Phi = Bd_{(0)} + Bd_{(1)}$. Let $\nu_0a_0 + \nu_1a_1 = \mu_0a_0 + \mu_1a_1$ with $\nu_i, \mu_i \in \mathbf{Z}$, $0 \leq \nu_0 \leq a_1-2$, $0 \leq \nu_1 \leq a_0-2$, $0 \leq \mu_0 \leq a_1-2$ and $0 \leq \mu_1 \leq a_0-2$. Then we may assume that $\nu_0 \geq \mu_0$ and $\nu_1 \leq \mu_1$. Now we have $(\nu_0 - \mu_0)a_0 = (\nu_1 - \mu_1)a_1$. In view of $(a_0, a_1)=1$, we obtain

$$\nu_0 - \mu_0 = r_1a_1 \text{ with } r_1 \geq 0 \text{ and } \mu_1 - \nu_1 = r_0a_0 \text{ with } r_0 \geq 0.$$

Since we have $\nu_0 - \mu_0 \leq a_1-2$ and $\mu_1 - \nu_1 \leq a_0-2$, we get $r_1=0$ and $r_0=0$, which imply that $\nu_0 = \mu_0$ and $\nu_1 = \mu_1$. Hence we have

$$T_X^1(\nu) = \{0\} \text{ or } \dim_k T_X^1(\nu) = 1,$$

which implies that

$$\begin{aligned} \mathbf{D}_X(k[\epsilon]) &= \bigoplus_{0 \leq \nu_0 \leq a_1-2} \bigoplus_{0 \leq \nu_1 \leq a_0-2} T_X^1(\nu_0a_0 + \nu_1a_1 - a_0a_1) \\ &= \bigoplus_{0 \leq \nu_0 \leq a_1-2} \bigoplus_{0 \leq \nu_1 \leq a_0-2} k\Phi(\theta_{\nu_0\nu_1}). \end{aligned}$$

Lastly we show that $\Phi(\theta_{\nu_0\nu_1}) \neq 0$ for $0 \leq \nu_0 \leq a_1-2$ and $0 \leq \nu_1 \leq a_0-2$. Assume that $\Phi(\theta_{\nu_0\nu_1}) = 0$. Since $d_{(0)}$ (resp. $d_{(1)}$) is of weighted degree $(a_1-1)a_0 - a_0a_1$ (resp. $(a_0-1)a_1 - a_0a_1$), we must have

$$\nu_0a_0 + \nu_1a_1 = (\mu_0 + a_1 - 1)a_0 + \mu_1a_1 \text{ with non-negative integers } \mu_0, \mu_1$$

or

$$\nu_0a_0 + \nu_1a_1 = x_0a_0 + (x_1 + a_0 - 1)a_1 \text{ with non-negative integers } x_0, x_1.$$

If $\nu_0a_0 + \nu_1a_1 = (\mu_0 + a_1 - 1)a_0 + \mu_1a_1$ holds, then $\nu_0 \geq \mu_0 + a_1 - 1$ or $\nu_1 \geq \mu_1$. The inequality $\nu_0 \geq \mu_0 + a_1 - 1$ contradicts $a_1 - 2 \geq \nu_0$. If $\nu_1 \geq \mu_1$, then

$$(\nu_1 - \mu_1)a_1 = (\mu_0 + a_1 - 1 - \nu_0)a_0,$$

which implies that $\nu_1 = \mu_1$, because of $\nu_1 \leq a_0-2$. Hence we get $\nu_0 = \mu_0 + a_1 - 1$, which contradicts $\nu_0 \leq a_1 - 2$. If $\nu_0a_0 + \nu_1a_1 = x_0a_0 + (x_1 + a_0 - 1)a_1$ holds, then similarly this contradicts $\nu_1 \leq a_0 - 2$. Hence we get $\Phi(\theta_{\nu_0\nu_1}) \neq 0$. *Q.E.D.*

By §1 and Proposition in §2 we have

$$\dim C_H = \dim_k \bigoplus_{\nu < 0} T_X^1(\nu) - 1.$$

By the above Lemma, we obtain

$$\begin{aligned} \dim C_H + 1 &= \#\{(\nu_0, \nu_1) \in \mathbf{N} \times \mathbf{N} \mid 0 \leq \nu_0 \leq a_1 - 2, \\ &\quad 0 \leq \nu_1 \leq a_0 - 2, \nu_0a_0 + \nu_1a_1 - a_0a_1 < 0\}. \end{aligned}$$

Theorem 2. If H is a numerical semigroup generated by a_0 and a_1 with $a_0 < a_1$, then we have

$$\begin{aligned}\dim C_H &= \frac{(a_0-1)(a_1-1)}{2} + a_0 + a_1 - \left[\frac{a_1}{a_0} \right] - 4 \\ &= g(H) + a_0 + a_1 - \left[\frac{a_1}{a_0} \right] - 4 = \frac{(a_0+1)(a_1+1)}{2} - \left[\frac{a_1}{a_0} \right] - 4,\end{aligned}$$

where $g(H)$ is the genus of H , i.e., the number of the complement $\mathbf{N} \setminus H$ of H in \mathbf{N} and $[]$ denotes the Gauss symbol.

Proof. In view of $\nu_0 a_0 + \nu_1 a_1 - a_0 a_1 < 0$, we have $\nu_0 < a_1 - \frac{\nu_1 a_1}{a_0}$. Since we have $\nu_1 \leq a_0 - 2$, we obtain

$$a_1 - \frac{\nu_1 a_1}{a_0} \geq a_1 + \frac{a_1}{a_0} (2 - a_0) = \frac{2a_1}{a_0} > 0.$$

If $1 \leq \nu_1 \leq a_0 - 2$, then we have

$$\left[a_1 - \frac{\nu_1 a_1}{a_0} \right] = a_1 + \left[-\frac{\nu_1 a_1}{a_0} \right] \leq a_1 - 1 - \left[\frac{a_1}{a_0} \right] \leq a_1 - 2.$$

Hence we have

$$\begin{aligned}\dim C_H &= -1 + \sum_{\nu_1=0}^{a_0-2} \# \{ \nu_0 \in \mathbf{N} \mid 0 \leq \nu_0 \leq a_1 - 2, \nu_0 < a_1 - \frac{\nu_1 a_1}{a_0} \} \\ &= -1 + a_1 - 1 + \sum_{\nu_1=1}^{a_0-2} \# \left\{ \nu_0 \in \mathbf{N} \mid 0 \leq \nu_0 \leq \left[a_1 - \frac{\nu_1 a_1}{a_0} \right] \right\} \\ &= -2 + a_1 + \sum_{\nu_1=1}^{a_0-2} \left(\left[a_1 - \frac{\nu_1 a_1}{a_0} \right] + 1 \right) \\ &= -2 + a_1 + a_0 - 2 + \sum_{\nu_1=1}^{a_0-2} \left(a_1 + \left[-\frac{\nu_1 a_1}{a_0} \right] \right) \\ &= -4 + a_0 + a_1 + (a_0 - 2)a_1 + \sum_{\nu_1=1}^{a_0-2} \left(-\left[\frac{\nu_1 a_1}{a_0} \right] - 1 \right) \\ &= -2 + a_0 a_1 - a_1 - \sum_{\nu_1=1}^{a_0-1} \left[\frac{\nu_1 a_1}{a_0} \right] + \left[\frac{(a_0-1)a_1}{a_0} \right] \\ &= -3 + a_0 a_1 - \left[\frac{a_1}{a_0} \right] - \sum_{\nu_1=1}^{a_0-1} \left[\frac{\nu_1 a_1}{a_0} \right].\end{aligned}$$

Moreover, we have

$$\begin{aligned}\sum_{\nu_1=1}^{a_0-1} \left[\frac{\nu_1 a_1}{a_0} \right] &= \sum_{\nu_1=1}^{a_0-1} \left[\frac{(a_0 - \nu_1)a_1}{a_0} \right] = \sum_{\nu_1=1}^{a_0-1} \left(a_1 + \left[-\frac{\nu_1 a_1}{a_0} \right] \right) \\ &= (a_0 - 1)a_1 + \sum_{\nu_1=1}^{a_0-1} \left(-\left[\frac{\nu_1 a_1}{a_0} \right] - 1 \right) = (a_0 - 1)a_1 - (a_0 - 1) - \sum_{\nu_1=1}^{a_0-1} \left[\frac{\nu_1 a_1}{a_0} \right],\end{aligned}$$

which implies that

$$\sum_{\nu_1=1}^{a_0-1} \left[\frac{\nu_1 a_1}{a_0} \right] = \frac{(a_0-1)(a_1-1)}{2}.$$

Hence we obtain

$$\begin{aligned}\dim C_H &= -3 + a_0 a_1 - \left[\frac{a_1}{a_0} \right] - \frac{(a_0-1)(a_1-1)}{2} \\ &= -4 - \left[\frac{a_1}{a_0} \right] + \frac{(a_0-1)(a_1-1)}{2} + a_0 + a_1. \quad Q.E.D.\end{aligned}$$

Lastly we show that the Rauch's result⁴⁾ is incorrect. It is the following:

Let T^g be the Teichmüller space of genus $g \geq 2$. Then the sublocus $T_{\nu, \nu+1}^g$ of T^g consisting of Riemann surfaces having a Weierstrass point x for which ν and $\nu+1$ are respectively the first and second non-gap is a subvariety of T^g of dimension $2g + \nu - 4$.

In fact, let $g = \frac{(\nu-1)\nu}{2}$ with $\nu \geq 4$. Let P be a Weierstrass point on C of genus g whose first and second non-gaps are ν and $\nu+1$ respectively. Then we have $H(P) \supseteq \langle \nu, \nu+1 \rangle$, where $\langle \nu, \nu+1 \rangle$ is the subsemigroup of \mathbf{N} generated by ν and $\nu+1$. Since we have

$$g = \# \mathbf{N} \setminus H(P) \leq \# \mathbf{N} \setminus \langle \nu, \nu+1 \rangle = \frac{(\nu-1)\nu}{2} = g,$$

which implies that $H(P) = \langle \nu, \nu+1 \rangle$. Hence by Theorem 2 we have

$$\dim T_{\nu, \nu+1}^g = \dim C_{\langle \nu, \nu+1 \rangle} = g + \nu + \nu + 1 - \left[\frac{\nu+1}{\nu} \right] - 4 = g + 2\nu - 4.$$

We note that Coppens⁵⁾ also showed $\dim C_{\langle \nu, \nu+1 \rangle} = g + 2\nu - 4$. But by the Rauch's result we get

$$g + 2\nu - 4 = \dim T_{\nu, \nu+1}^g = 2g + \nu - 4,$$

which implies that

$$\frac{(\nu-1)\nu}{2} = g = \nu.$$

Hence we get $\nu = 3$ or $\nu = 0$. This is a contradiction.

References

- 1) M. Schlessinger: Functors of Artin rings, Trans. Amer. Math. Soc. **130** (1968) 208-222.
- 2) H.C. Pinkham: Deformations of algebraic varieties with \mathbf{G}_m -action, Astérisque **20** (1974) 1-131.
- 3) J. Komeda: On the existence of Weierstrass points with a certain semigroup generated by 4 elements, Tsukuba J. Math. **6** (1982) 237-270.
- 4) H.E. Rauch: Weierstrass points, branch points, and the moduli of Riemann surfaces, Comm. Pure Appl. Math. **12** (1959) 543-560.
- 5) M. Coppens: Weierstrass points with two prescribed non-gaps, Pacific J. Math. **131** (1988) 71-104.