On Non-primitive Weierstrass Gap Sequences of Genus 8

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Abstract

Let L be a *gap sequence*, i.e., a finite subset of the additive semigroup $\mathbf{N}$ of non-negative integers whose complement $\mathbf{N}\backslash L$ in $\mathbf{N}$ forms a subsemigroup of $\mathbf{N}$. Then the order of the set L is called its *genus*. If twice the smallest positive integer of $\mathbf{N}\backslash L$ is larger than the largest integer of L, we say that L is *primitive*. It is known that any gap sequence L of genus ≤ 7 (resp. any primitive gap sequence L of genus 8) is *Weierstrass*[1], i.e., there exists a pointed complete non-singular irreducible algebraic curve (C, P) over the field $\mathbf{C}$ of complex numbers such that $\mathbf{N}\backslash L$ is the set of integers which are pole orders at P of regular functions on $C\backslash\{P\}$. In this paper we investigate non-primitive gap sequences of genus 8. The result is that every non-primitive gap sequence except four sequences is Weierstrass.

§1. On non-primitive gap sequences of genus 8

For a gap sequence $L=\{l_0<l_1<\cdots<l_{g-1}\}$ of genus g, $H(L)$ denotes the complement of L in $\mathbf{N}$, which is a subsemigroup of $\mathbf{N}$. Let $M(L)$ be the minimal set of generators for $H(L)$. We set

$$\alpha(L)=(\alpha_0(L), \alpha_1(L), \cdots, \alpha_{g-1}(L)),$$

where $\alpha_i(L)=l_i-i-1$ for any $i=0, 1, \cdots, g-1$. Then the following table shows all non-primitive gap sequences L of genus 8.

	L	$M(L)$	$\alpha(L)$
(1)	$\{1, 3, 5, 7, 9, 11, 13, 15\}$	$\{2, 17\}$	$(0, 1, 2, 3, 4, 5, 6, 7)$
(2)	$\{1, 2, 4, 5, 7, 8, 11, 14\}$	$\{3, 10, 17\}$	$(0, 0, 1, 1, 2, 2, 4, 6)$
(3)	$\{1, 2, 4, 5, 7, 8, 10, 13\}$	$\{3, 11, 16\}$	$(0, 0, 1, 1, 2, 2, 3, 5)$
(4)	$\{1, 2, 4, 5, 7, 8, 10, 11\}$	$\{3, 13, 14\}$	$(0, 0, 1, 1, 2, 2, 3, 3)$
(5)	$\{1, 2, 3, 5, 7, 9, 11, 15\}$	$\{4, 6, 13\}$	$(0^3, 1, 2, 3, 4, 7)$
(6)	$\{1, 2, 3, 5, 7, 9, 11, 13\}$	$\{4, 6, 15, 17\}$	$(0^3, 1, 2, 3, 4, 5)$
(7)	$\{1, 2, 3, 5, 6, 9, 10, 13\}$	$\{4, 7, 17\}$	$(0^3, 1, 1, 3, 3, 5)$
(8)	$\{1, 2, 3, 5, 6, 7, 11, 15\}$	$\{4, 9, 10\}$	$(0^3, 1^3, 4, 7)$
(9)	$\{1, 2, 3, 5, 6, 7, 10, 14\}$	$\{4, 9, 11\}$	$(0^3, 1^3, 3, 6)$
(10)	$\{1, 2, 3, 5, 6, 7, 10, 11\}$	$\{4, 9, 14, 15\}$	$(0^3, 1^3, 3, 3)$
(11)	$\{1, 2, 3, 5, 6, 7, 9, 13\}$	$\{4, 10, 11, 17\}$	$(0^3, 1^3, 2, 5)$
(12)	$\{1, 2, 3, 5, 6, 7, 9, 11\}$	$\{4, 10, 13, 15\}$	$(0^3, 1^3, 2, 3)$

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(13)	$\{1, 2, 3, 5, 6, 7, 9, 10\}$	$\{4, 11, 13, 14\}$	$(0^3, 1^3, 2, 2)$
(14)	$\{1, 2, 3, 4, 7, 8, 9, 14\}$	$\{5, 6, 13\}$	$(0^4, 2^3, 6)$
(15)	$\{1, 2, 3, 4, 7, 8, 9, 13\}$	$\{5, 6, 14\}$	$(0^4, 2^3, 5)$
(16)	$\{1, 2, 3, 4, 6, 8, 11, 13\}$	$\{5, 7, 9\}$	$(0^4, 1, 2, 4, 5)$
(17)	$\{1, 2, 3, 4, 6, 8, 9, 13\}$	$\{5, 7, 11\}$	$(0^4, 1, 2, 2, 5)$
(18)	$\{1, 2, 3, 4, 6, 8, 9, 11\}$	$\{5, 7, 13, 16\}$	$(0^4, 1, 2, 2, 3)$
(19)	$\{1, 2, 3, 4, 6, 7, 11, 12\}$	$\{5, 8, 9\}$	$(0^4, 1, 1, 4, 4)$
(20)	$\{1, 2, 3, 4, 6, 7, 9, 14\}$	$\{5, 8, 11, 12\}$	$(0^4, 1, 1, 2, 6)$
(21)	$\{1, 2, 3, 4, 6, 7, 9, 12\}$	$\{5, 8, 11, 14, 17\}$	$(0^4, 1, 1, 2, 4)$
(22)	$\{1, 2, 3, 4, 6, 7, 9, 11\}$	$\{5, 8, 12, 14\}$	$(0^4, 1, 1, 2, 3)$
(23)	$\{1, 2, 3, 4, 6, 7, 8, 13\}$	$\{5, 9, 11, 12\}$	$(0^4, 1^3, 5)$
(24)	$\{1, 2, 3, 4, 6, 7, 8, 12\}$	$\{5, 9, 11, 13, 17\}$	$(0^4, 1^3, 4)$
(25)	$\{1, 2, 3, 4, 6, 7, 8, 11\}$	$\{5, 9, 12, 13, 16\}$	$(0^4, 1^3, 3)$
(26)	$\{1, 2, 3, 4, 5, 8, 9, 15\}$	$\{6, 7, 10, 11\}$	$(0^5, 2, 2, 7)$
(27)	$\{1, 2, 3, 4, 5, 7, 11, 13\}$	$\{6, 8, 9, 10\}$	$(0^5, 1, 4, 5)$
(28)	$\{1, 2, 3, 4, 5, 7, 10, 13\}$	$\{6, 8, 9, 11\}$	$(0^5, 1, 3, 5)$
(29)	$\{1, 2, 3, 4, 5, 7, 9, 15\}$	$\{6, 8, 10, 11, 13\}$	$(0^5, 1, 2, 7)$
(30)	$\{1, 2, 3, 4, 5, 7, 9, 13\}$	$\{6, 8, 10, 11, 15\}$	$(0^5, 1, 2, 5)$
(31)	$\{1, 2, 3, 4, 5, 7, 8, 14\}$	$\{6, 9, 10, 11, 13\}$	$(0^5, 1, 1, 6)$
(32)	$\{1, 2, 3, 4, 5, 7, 8, 13\}$	$\{6, 9, 10, 11, 14\}$	$(0^5, 1, 1, 5)$
(33)	$\{1, 2, 3, 4, 5, 6, 8, 15\}$	$\{7, 9, 10, 11, 12, 13\}$	$(0^6, 1, 7)$

If the smallest positive integer of $H(L)$ is less than or equal to 5, then L is Weierstrass[2~4]. Hence the first twenty five gap sequences of the above table are Weierstrass.

§2. On the gap sequence $\{1, 2, 3, 4, 5, 8, 9, 15\}$

Let L be a gap sequence with $M(L)=\{a_0<a_1<\cdots<a_{n-1}\}$. Hereafter we use the following notation:

$$c_i=\mathrm{Min}\{c\in\mathbf{N}>0 \mid ca_i\in<a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n-1}>\}$$

for all $i=0, 1, \cdots, n-1$, where for positive integers $b_1, \cdots, b_m$, $<b_1, \cdots, b_m>$ denotes the subsemigroup of $\mathbf{N}$ generated by $b_1, \cdots, b_m$.

Let $L=\{1, 2, 3, 4, 5, 8, 9, 15\}$. Then $M(L)=\{6, 7, 10, 11\}$. We set $a_0=6$, $a_1=7$, $a_2=10$ and $a_3=11$. Then we have the following relations:

$$3a_0=a_1+a_3,\ 3a_1=a_2+a_3,\ 2a_2=a_0+2a_1 \text{ and } 2a_3=2a_0+a_2,$$

which imply that $c_0=3$, $c_1=3$, $c_2=2$ and $c_3=2$. Moreover, we have

$$\begin{vmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ -1 & -2 & 2 \end{vmatrix}=11=a_3.$$

Hence $H(L)$ is 1-neat, which implies that L is Weierstrass[5].

§3. On the gap sequence $\{1, 2, 3, 4, 5, 7, 8, 14\}$

In this section we shall show that $L=\{1, 2, 3, 4, 5, 7, 8, 14\}$ is Weierstrass, i.e., there exists a pointed curve (C, P) such that $L=\mathbf{N}\backslash H(P)$, where $H(P)$ is the set of integers which are pole orders at P of regular functions on $C\backslash\{P\}$.

Let E be an elliptic curve over $\mathbf{C}$ with the origin Q'. Let P_1' be a point of E such that $P_1'\neq Q'$ and $2P_1'=Q'$, i.e., $2P_1'\sim 2Q'$. Moreover, P_2' denotes a point of E such that $P_2'\neq Q'$, $P_2'\neq P_1'$ and $-5P_1'=3P_2'$, i.e., $5P_1'+3P_2'\sim 8Q'$. Take $z\in\mathbf{K}(E)$ such that div $(z)=5P_1'+3P_2'-8Q'$, where $\mathbf{K}(E)$ denotes the function field of E. Let $\pi: C\longrightarrow E$ be the surjective morphism corresponding to the inclusion $\mathbf{K}(E)\subset\mathbf{K}(E)(z^{1/8})=\mathbf{K}(C)$. Let $y\in\mathbf{K}(C)$ and $\sigma\in \mathrm{Aut}(\mathbf{K}(C)/\mathbf{K}(E))$ such that $\sigma(y)=\zeta_8 y$ and $\mathrm{div}_E(y^8)=5P_1'+3P_2'-8Q'$, where ζ_8 is a primitive 8-th root of unity. Then there are only two ramification points P_1 and P_2 over P_1' and P_2' respectively and the ramification indices are 8. Hence by the Riemann-Hurwitz relation the genus of C is 8.

Theorem 3.1. *We have* $\mathbf{N}\backslash H(P_1)=\{1, 2, 3, 4, 5, 7, 8, 14\}$.

Proof. We have

$$\mathrm{div}(y)=5P_1+3P_2-\pi^*(Q')$$

and

$$\mathrm{div}(dy)=4P_1+2P_2-2\pi^*(Q')+\sum_{i=1}^{3}\pi^*(R_i'),$$

where R_i''s are points of E which are distinct from P_1', P_2' and Q'. For any $f\in\mathbf{K}(E)$, we set $\mathrm{div}_E(f)=\sum_{P'\in E} n(P')P'$. Then for any $r\in\mathbf{N}$ we obtain

$$\mathrm{div}\left(\frac{fdy}{y^{1-r}}\right)=\{8n(P'_1)+4+5(r-1)\}P_1+\{8n(P'_2)+2+3(r-1)\}P_2$$
$$+\{n(Q')-r-1\}\pi^*(Q')+\sum_{i=1}^{3}\{n(R_i')+1\}\pi^*(R_i')+\sum_{P'\in S} n(P')\pi^*(P'),$$

where S is the set of points $P'\in E$ except P'_1, P'_2, Q' and R_i''s. We set

$$D_0'=-P_1'-P_2'-Q'+\sum_{i=1}^{3} R_i',\ D_1'=-2Q'+\sum_{i=1}^{3} R_i',$$
$$D_2'=-3Q'+P_1'+\sum_{i=1}^{3} R_i',\ D_3'=-4Q'+P_1'+P_2'+\sum_{i=1}^{3} R_i',$$
$$D_4'=-5Q'+2P_1'+P_2'+\sum_{i=1}^{3} R_i',\ D_5'=-6Q'+3P_1'+P_2'+\sum_{i=1}^{3} R_i',$$
$$D_6'=-7Q'+3P_1'+2P_2'+\sum_{i=1}^{3} R_i',$$
$$D_7'=-8Q'+4P_1'+2P_2'+\sum_{i=1}^{3} R_i'.$$

Then for each $r=0, 1, \cdots, 7$, $f\in L(D_r')$ implies that fdy/y^{1-r} is a holomorphic differential form on C, where

$$L(D'_r)=\{f\in\mathbf{K}(E)\mid \mathrm{div}_E(f)\geq -D'_r\}.$$

Since we have

$$\sigma\left(\frac{dy}{y}\right)=\frac{d\sigma y}{\sigma y}=\frac{d\zeta_8 y}{\zeta_8 y}=\frac{dy}{y},$$

the form dy/y is regarded as a differential form on E. Hence there exists $f\in\mathbf{K}(E)$ such that fdy/y is holomorphic. Then we must have

$$\mathrm{div}_E(f)=P'_1+P'_2+Q'-\sum_{i=1}^{3}R'_i, \text{ i.e., } l(D'_0)=1,$$

where for any divisor D we denote by $l(D)$ the dimension of the $\mathbf{C}$-vector space $L(D)$. Moreover, we have $l(D'_r)=1$ for all $r=1, 2, \cdots, 7$, because of $\deg(D'_r)=1$ for all $r=1, 2, \cdots, 7$. First we will show that $l(D'_1-P'_1)=0$. If $l(D'_1-P'_1)>0$, then we have

$$-2Q'+\sum_{i=1}^{3}R'_i-P'_1\sim D'_1-P'_1\sim 0\sim D'_0=-P'_1-P'_2-Q'+\sum_{i=1}^{3}R'_i,$$

which implies that $P'_2\sim Q'$. This is a contradiction. If $l(D'_2-P'_1)>0$, then we have

$$-3Q'+\sum_{i=1}^{3}R'_i=D'_2-P'_1\sim D'_0=-P'_1-P'_2-Q'+\sum_{i=1}^{3}R'_i,$$

which implies that $P'_1+P'_2\sim 2Q'\sim 2P'_1$. This is a contradiction. If $l(D'_3-P'_1)>0$, then we have

$$-4Q'+P'_2+\sum_{i=1}^{3}R'_i=D'_3-P'_1\sim D'_0=-P'_1-P'_2-Q'+\sum_{i=1}^{3}R'_i,$$

which implies that $P'_1+2P'_2\sim 3Q'$. Since we have

$$5P'_1+3P'_2\sim 8Q'\sim 4Q'+4P'_1,$$

we obtain

$$P'_1+2P'_2+Q'\sim 4Q'\sim P'_1+3P'_2,$$

which implies that $Q'\sim P'_2$. This is a contradiction. If $l(D'_4-P'_1)>0$, then we have

$$-5Q'+P'_1+P'_2+\sum_{i=1}^{3}R'_i=D'_4-P'_1\sim D'_0=-P'_1-P'_2-Q'+\sum_{i=1}^{3}R'_i,$$

which implies that

$$2P'_1+2P'_2\sim 4Q'\sim P'_1+3P'_2.$$

This is a contradiction. If $l(D'_5-P'_1)>0$, then we have

$$-6Q'+2P'_1+P'_2+\sum_{i=1}^{3}R'_i=D'_5-P'_1\sim D'_0=-P'_1-P'_2-Q'+\sum_{i=1}^{3}R'_i,$$

which implies that

$$3P'_1+2P'_2\sim 5Q'\sim Q'+P'_1+3P'_2.$$

Hence we have

$$2Q' \sim 2P_1' \sim Q' + P_2'.$$

This is a contradiction. Now we have

$$6Q' \sim 2Q' + P_1' + 3P_2' \sim 2P_1' + P_1' + 3P_2' = 3P_1' + 3P_2'.$$

Hence we get

$$D_6' - P_1' = -7Q' + 2P_1' + 2P_2' + \sum_{i=1}^{3} R_i' \sim Q' - P_1' - P_2' + \sum_{i=1}^{3} R_i' = D_0' \sim 0,$$

which implies that $l(D_6') = l(D_6' - P_1') = 1$. If $l(D_7' - P_1') > 0$, then we have

$$-8Q' + 3P_1' + 2P_2' + \sum_{i=1}^{3} R_i' = D_7' - P_1' \sim D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that

$$4P_1' + 3P_2' \sim 7Q' \sim 4P_1' + 3Q'.$$

Hence we get

$$3P_2' + Q' \sim 4Q' \sim P_1' + 3P_2'.$$

This is a contradiction. By the above we have

$$l(D_i' - P_1') = 0 \text{ for all } i \neq 6 \text{ and } l(D_6' - P_1') = 1,\ l(D_6' - 2P_1') = 0.$$

For each $r = 0, 1, \cdots, 7$ we take a non-zero element $f_r \in L(D_r')$ and we set $\phi_r = f_r dy/y^{1-r}$. Then by the above we see the following:

$$\begin{aligned}
&\mathrm{ord}_{P_1}(\phi_0) = 8 - 1 = 7 = 8 - 1,\ \mathrm{ord}_{P_1}(\phi_1) = 0 + 4 = 5 - 1,\\
&\mathrm{ord}_{P_1}(\phi_2) = -8 + 9 = 1 = 2 - 1,\ \mathrm{ord}_{P_1}(\phi_3) = -8 + 14 = 6 = 7 - 1,\\
&\mathrm{ord}_{P_1}(\phi_4) = -16 + 19 = 3 = 4 - 1,\ \mathrm{ord}_{P_1}(\phi_5) = -24 + 24 = 0 = 1 - 1,\\
&\mathrm{ord}_{P_1}(\phi_6) = -16 + 29 = 13 = 14 - 1,\\
&\mathrm{ord}_{P_1}(\phi_7) = -32 + 34 = 2 = 3 - 1.
\end{aligned}$$

We note that $n \in \mathbf{N} \backslash H(P_1)$ if and only if there exists a holomorphic differential form ϕ on C such that $\mathrm{ord}_{P_1}(\phi) = n - 1$. Hence we obtain $\mathbf{N} \backslash H(P_1) = \{1, 2, 3, 4, 5, 7, 8, 14\}$. *Q.E.D.*

§4. On the gap sequence $\{1, 2, 3, 4, 5, 7, 11, 13\}$

For a gap sequence L, let $M(L) = \{a_0 < a_1 < \cdots < a_{n-1}\}$. We denote by φ_L the $\mathbf{C}$-algebra homomorphism from $\mathbf{C}[X] = \mathbf{C}[X_0, X_1, \cdots, X_{n-1}]$ to $\mathbf{C}[L] = \mathbf{C}[t^h]_{h \in H(L)}$ defined by sending X_i to t^{a_i} for all i. Let I_L be the kernel of φ_L. In this case, we define a weight on $\mathbf{C}[X]$ through φ_L as follows: For any i, the weighted degree of X_i is a_i and for any non-zero element c of $\mathbf{C}$, the weighted degree of c is zero. For any monomial f in $\mathbf{C}[X]$, $w(f)$ denotes the weighted degree of f. In this section we shall show that $L = \{1, 2, 3, 4, 5, 7, 11, 13\}$ is Weierstrass.

Lemma 4.1. *The ideal I_L is generated by*

$$X_1^2 - X_0X_3,\ X_3^2 - X_0^2X_1,\ X_2^2 - X_1X_3 \text{ and } X_0^3 - X_1X_3.$$

Proof. We have $M(H)=\{6, 8, 9, 10\}$. Hence $a_0=6$, $a_1=8$, $a_2=9$ and $a_3=10$. Then we obtain relations

$$2a_1 = a_0 + a_3,\ 2a_3 = 2a_0 + a_1,\ 2a_2 = a_1 + a_3 \text{ and } 3a_0 = a_1 + a_3,$$

which imply that $c_1=c_3=c_2=2$ and $c_0=3$. Hence the ideal I_L contains

$$X_1^2 - X_0X_3,\ X_3^2 - X_0^2X_1,\ X_2^2 - X_1X_3 \text{ and } X_0^3 - X_1X_3.$$

Let J be the ideal generated by the above four elements. We may take as generators for the ideal I_L the following type:

$$F = \prod_i X_i^{\nu_i} - \prod_i X_i^{\mu_i},\ \nu_i\mu_i = 0 \text{ for all } i^{6)}.$$

We set $L=\prod_i X_i^{\nu_i}$. Then we may assume that F is one of the following types:

(1) $L=X_0L_0$ and $R=X_iX_jR_0$, $1 \leq i \leq j$, where L_0 and R_0 are elements of $\mathbf{C}[X]$ and R_0 does not contain X_0.

(2) Neither L nor R contains X_0, i.e., $L=X_lX_mL_0$, $1 \leq l \leq m$ and $R=X_iX_jR_0$, $1 \leq i \leq j$, where neither L_0 nor R_0 contains X_0.

We consider the case (1). If $(i, j)=(1, 1)$, then

$$\begin{aligned} F &= X_0L_0 - X_1^2R_0 = X_0L_0 - (X_1^2 - X_0X_3)R_0 - X_0X_3R_0 \\ &\equiv X_0L_0 - X_0X_3R_0 \bmod J. \end{aligned}$$

Since $J \subseteq I_L$ and $X_0L_0 - X_1^2R_0 \in I_L$, we have $X_0L_0 - R_0X_0X_3 \in I_L$. Hence we obtain $L_0 - R_0X_3 \in I_L$, because I_L is a prime ideal which does not contain X_0. Therefore we may decrease the weighted degree of F. Let $(i, j)=(1, 2)$. If L_0 contains either X_1 or X_2, we may decrease the weighted degree of F. Hence we may assume that

$$\begin{aligned} F &= X_0X_3L_1 - X_1X_2R_0 = (X_0X_3 - X_1^2)L_1 + X_1^2L_1 - X_1X_2R_0 \\ &\equiv X_1^2L_1 - X_1X_2R_0 \bmod J. \end{aligned}$$

Therefore we may decrease the weighted degree of F. If $(i, j)=(1, 3)$, then

$$\begin{aligned} F &= X_0L_0 - X_1X_3R_0 = X_0L_0 - (X_1X_3 - X_0^3)R_0 - X_0^3R_0 \\ &\equiv X_0L_0 - X_0^3R_0 \bmod J. \end{aligned}$$

Hence we may decrease the weighted degree of F. If $(i, j)=(2, 2)$, then

$$\begin{aligned} F &= X_0L_0 - X_2^2R_0 = X_0L_0 - (X_2^2 - X_1X_3)R_0 - (X_1X_3 - X_0^3)R_0 - X_0^3R_0 \\ &\equiv X_0L_0 - X_0^3R_0 \bmod J. \end{aligned}$$

Hence we may decrease the weighted degree of F. If $(i, j)=(2, 3)$, then we may assume that

$$\begin{aligned} F &= X_0X_1L_1 - X_2^2R_0 = X_0X_1L_1 - (X_2^2 - X_1X_3)R_0 - X_1X_3R_0 \\ &\equiv X_0X_1L_1 - X_1X_3R_0 \bmod J. \end{aligned}$$

Hence we may decrease the weighted degree of F. If $(i, j)=(3, 3)$, then

$$F = X_0L_0 - X_3^2R_0 = X_0L_0 - (X_3^2 - X_0^2X_1)R_0 - X_0^2X_1R_0$$
$$\equiv X_0L_0 - X_0^2X_1R_0 \bmod J.$$

Hence we may decrease the weighted degree of F.

Next we consider the case (2). If (i, j) (resp. (l, m))$=(1, 1), (1, 3), (2, 2)$ or $(3, 3)$, then this case is reduced to the case (1), because J contains

$$X_1^2 - X_0X_3,\ X_0^3 - X_1X_3,\ X_2^2 - X_1X_3 \text{ and } X_3^2 - X_0^2X_1.$$

Hence we may assume that (i, j) (resp. (l, m)) is one of the following: $(1, 2), (2, 3)$. In the above cases we may decrease the weighted degree of F. Therefore we have $F \in J$. Hence the ideal I_L is generated by

$$X_1^2 - X_0X_3,\ X_3^2 - X_0^2X_1,\ X_2^2 - X_1X_3 \text{ and } X_0^3 - X_1X_3.$$

Q.E.D.

Let $\mathbf{Z}$ be the additive group of integers. For any $i=1, \cdots, m$ we denote by e_i the vector in $\mathbf{Z}^m$ whose i-th component is equal to 1 and whose j-th component is equal to 0 if $j \neq i$. A subsemigroup S of $\mathbf{Z}^m$ is said to be *saturated* if the condition $nr \in S$, where n is a positive integer and r is an element of $\mathbf{Z}^m$, implies $r \in S$. Assume that the semigroup S is generated by $b_1, b_2, \cdots, b_q$. Then S is saturated if and only if

$$\sum_{i=1}^{q} \mathbf{R}_+ b_i \cap \mathbf{Z}^m = \sum_{i=1}^{q} \mathbf{N} b_i = S,$$

where $\mathbf{R}_+$ denotes the set of non-negative real numbers.

Let S be the subsemigroup of $\mathbf{Z}^4$ generated by $b_1, b_2, \cdots, b_7$, where

$$b_i = e_i \text{ for } i=1, 2, 3, 4,\ b_5=(1, 1, -1, 0),\ b_6=(1, 0, -1, 1) \text{ and } b_7=(1, 0, 0, 1).$$

Then we see the following:

Lemma 4.2. *The subsemigroup S of $\mathbf{Z}^4$ is saturated.*

Proof. Take $p=(p_1, p_2, p_3, p_4) \in \sum_{i=1}^{7} \mathbf{R}_+ b_i \cap \mathbf{Z}^4$. Then it suffices to show that $p \in S$. Moreover, we may assume that $p = \sum_{i=1}^{7} m_i b_i$ with $0 \leq m_i < 1$, all i. Then we obtain

$$p_1 = m_1 + m_5 + m_6 + m_7 \geq 0,\ p_2 = m_2 + m_5 \geq 0,$$
$$p_3 = m_3 - m_5 - m_6 \geq -1,\ p_4 = m_4 + m_6 + m_7 \geq 0.$$

To prove that S is saturated it suffices to show that $p \in S$ if $p_3 = -1$. Let $p_3 = -1$, i.e., $m_3 + 1 = m_5 + m_6$, which implies that $m_5 > 0$ and $m_6 > 0$. Hence we may assume that

$$p_1 = 1,\ p_2 = 1 \text{ and } p_4 = 1.$$

Therefore $p = (1, 1, -1, 1) = b_5 + b_4 \in S$. *Q.E.D.*

Now we have the following relations:

$$(c_{21} + c_{31})a_1 = c_{10}a_0 + c_{13}a_3,\ (c_{03} + c_{23})a_3 = c_{30}a_0 + c_{31}a_1,$$
$$c_2a_2 = c_{21}a_1 + c_{23}a_3,\ c_{21}a_1 + c_{23}a_3 = (c_{30} + c_{10})a_0,$$

where we set

$$c_{21}=c_{31}=c_{13}=c_{23}=c_{10}=1,\ c_2=2 \text{ and } c_{30}=2.$$

Hence we set

$$g_1=X_1^{c_{21}},\ g_2=X_1^{c_{31}},\ g_3=X_0^{c_{10}},\ g_4=X_3^{c_{23}},$$
$$g_5=X_3^{c_{13}},\ g_6=X_0^{c_{30}} \text{ and } g_7=X_2^{c_2}.$$

Let

$$\pi: \mathbf{C}[Y]=\mathbf{C}[Y_1,\cdots,Y_7]\longrightarrow \mathbf{C}[S]=\mathbf{C}[T^s]_{s\in S}$$
$$(\text{resp. } \eta: \mathbf{C}[Y]\longrightarrow \mathbf{C}[X]=\mathbf{C}[X_0,X_1,X_2,X_3])$$

be the $\mathbf{C}$-algebra homomorphism defined by $\pi(Y_i)=T^{b_i}$ (resp. $\eta(Y_i)=g_i$).

Lemma 4.3. *The ideal I_L is generated by the elements of the set $\eta(\mathrm{Ker}\ \pi)$.*

Proof. Let

$$\zeta: \mathbf{C}[\mathbf{N}^4]=\mathbf{C}[t_1,t_2,t_3,t_4]\longrightarrow \mathbf{C}[L]$$

be the $\mathbf{C}$-algebra homomorphism defined by $\zeta(t_i)=t^{w(g_i)}$. Then ζ extends to $\zeta': \mathbf{C}[S]\longrightarrow \mathbf{C}[L]$, because we have

$$w(g_1g_2g_3^{-1})=c_{21}a_1+c_{31}a_1-c_{10}a_0=c_{13}a_3=w(g_5),$$
$$w(g_1g_4g_3^{-1})=c_{21}a_1+c_{23}a_3-c_{10}a_0=c_{30}a_0=w(g_6),$$
$$w(g_1g_4)=c_{21}a_1+c_{23}a_3=c_2a_2=w(g_7).$$

Moreover, by the above we obtain $\varphi_L\circ\eta=\zeta'\circ\pi$, which implies that $\eta(\mathrm{Ker}\ \pi)\subseteq \mathrm{Ker}\ \varphi_L=I_L$. Since we have

$$b_1+b_2=b_3+b_5,\ b_5+b_4=b_6+b_2,\ b_7=b_1+b_4 \text{ and } b_1+b_4=b_6+b_3,$$

Ker π contains

$$Y_1Y_2-Y_3Y_5,\ Y_5Y_4-Y_6Y_2,\ Y_7-Y_1Y_4 \text{ and } Y_1Y_4-Y_6Y_3,$$

which implies that $\eta(\mathrm{Ker}\ \pi)$ contains the generators for I_L because of Lemma 4.1. Hence we obtain our desired result. *Q.E.D.*

Theorem 4.4. *The gap sequence $L=\{1,2,3,4,5,7,11,13\}$ is Weierstrass.*

Proof. By Lemmas 4.2 and 4.3, L is Weiertrass[3]. *Q.E.D.*

§5. On the gap sequence $\{1,2,3,4,5,7,9,13\}$.

In this sectioin we shall show that $L=\{1,2,3,4,5,7,9,13\}$ is Weierstrass. This proof is similar to that of §4. But it is more complicated. In this case we note that $M(L)=\{6,8,10,11,15\}$. Hence

$$a_0=6,\ a_1=8,\ a_2=10,\ a_3=11 \text{ and } a_4=15.$$

Lemma 5.1. *The ideal I_L is generated by*

$$X_3X_4-X_0X_2^2,\ X_0X_4-X_2X_3,\ X_1X_4-X_0^2X_3,$$
$$X_2X_4-X_0X_1X_3,\ X_0^3-X_1X_2,\ X_1^2-X_0X_2,$$
$$X_2^2-X_0^2X_1,\ X_3^2-X_0X_1^2 \text{ and } X_4^2-X_0^2X_1X_2.$$

Proof. We obtain relations

$$a_3+a_4=a_0+2a_2,\ a_0+a_4=a_2+a_3,\ a_1+a_4=2a_0+a_3,$$
$$a_2+a_4=a_0+a_1+a_3,\ 3a_0=a_1+a_2,\ 2a_1=a_0+a_2,$$
$$2a_2=2a_0+a_1,\ 2a_3=a_0+2a_1 \text{ and } 2a_4=2a_0+a_1+a_2,$$

which imply that $c_0=3$ and $c_1=c_2=c_3=c_4=2$. Hence the ideal I_L contains all elements in the statement of Lemma 5.1. Let J be the ideal generated by the above nine elements. We may take as generators for the ideal I_L the following type:

$$F=\prod_i X_i^{\nu_i}-\prod_i X_i^{\mu_i},\ \nu_i\mu_i=0 \text{ for all } i.$$

We set $L=\prod_i X_i^{\nu_i}$ and $R=\prod_i X_i^{\mu_i}$. Then we may assume that F is one of the following types:

(1) $L=X_4X_lL_0$, $0\leq l\leq 4$ and $R=X_iX_jR_0$, $0\leq i\leq j\leq 3$, where R_0 does not contain X_4.

(2) Neither L nor R contains X_4.

We consider the case (1). If $l=0$, then we may assume that $(i,j)=(1,1)$, because I_L contains $X_4X_0-X_2X_3$. Hence we have

$$F=X_4X_0L_0-X_1^2R_0=X_4X_0L_0-(X_1^2-X_0X_2)R_0-X_0X_2R_0$$
$$\equiv X_4X_0L_0-X_0X_2R_0 \bmod J.$$

Therefore we may decrease the weighted degree of F. If $l=1$, then we may assume that $(i,j)=(2,2)$, because I_L contains $X_4X_1-X_0^2X_3$. Hence we have

$$F=X_4X_1L_0-X_2^2R_0=X_4X_1L_0-(X_2^2-X_0^2X_1)R_0-X_0^2X_1R_0$$
$$\equiv X_4X_1L_0-X_0^2X_1R_0 \bmod J.$$

Therefore we may decrease the weighted degree of F. If $l=2$, we have

$$F=X_4X_2L_0-X_iX_jR_0$$
$$=(X_4X_2-X_0X_1X_3)L_0+X_0X_1X_3L_0-X_iX_jR_0 \bmod J.$$

Therefore we may decrease the weighted degree of F. If $l=3$, then we may assume that $(i,j)=(1,1)$, because I_L contains $X_4X_3-X_0X_2^2$. Hence we have

$$F=X_4X_3L_0-X_1^2R_0$$
$$=(X_4X_3-X_0X_2^2)L_0-(X_1^2-X_0X_2)R_0+X_0X_2^2L_0-X_0X_2R_0$$
$$\equiv X_0X_2^2L_0-X_0X_2R_0 \bmod J.$$

Therefore we may decrease the weighted degree of F. If $l=4$, then we may assume that $(i,j)=(3,3)$, because I_L contains $X_4^2-X_0^2X_1X_2$. Hence we have

$$F=X_4^2L_0-X_3^2R_0$$
$$=(X_4^2-X_0^2X_1X_2)L_0-(X_3^2-X_0X_1^2)R_0+X_0^2X_1X_2L_0-X_0X_1^2R_0$$
$$\equiv X_0^2X_1X_2L_0-X_0X_1^2R_0 \bmod J.$$

Therefore we may decrease the weighted degree of F.

Next we consider the case (2). We set

$$L=X_l X_m L_0,\ l\leq m\leq 3 \text{ and } R=X_i X_j R_0,\ i\leq j\leq 3,$$

where neither L_0 nor R_0 contains X_4. We may assume that $l<i$. Let $l=0$. Then we have

$$F=X_0 X_m L_0-X_i X_j R_0,\ 1\leq i\leq j.$$

If $(i, j)\neq(1, 3)$, we may decrease the weighted degree of F. If $(i, j)=(1, 3)$, we have $F=X_0 X_m L_0-X_1 X_3 R_0$. Since I_L contains $X_0X_2-X_1^2$ and $X_0X_4-X_2X_3$, we may decrease the weighted degree of F. Let $l=1$. Then we have $F=X_1X_mL_0-X_iX_jR_0$, $2\leq i\leq j$. If $(i, j)=(2, 2)$, $(2, 4)$, $(3, 3)$ or $(4, 4)$, we may decrease the weighted degree of F. If $(i, j)=(2, 3)$ or $(3, 4)$, this case is reduced to the case $l=0$, because I_L contains $X_2X_3-X_0X_4$ and $X_3X_4-X_0X_2^2$. Let $l=2$. Then we have $F=X_2X_mL_0-X_iX_jR_0$, $3\leq i\leq j$. This case is reduced to the case $l=0$. Let $l=3$. Then we have

$$F=X_3X_mL_0-X_4^2R_0=X_3X_mL_0-(X_4^2-X_0^2X_1X_2)R_0-X_0^2X_1X_2R_0$$
$$\equiv X_3X_mL_0-X_0^2X_1X_2R_0 \mod J.$$

Since I_L contains $X_3^2-X_0X_1^2$ and $X_3X_4-X_0X_2^2$, we may decrease the weighted degree of F. *Q.E.D.*

Let S be the subsemigroup of $\mathbf{Z}^5$ generated by $b_1, b_2, \cdots, b_9$, where $b_i=e_i$ for $1\leq i\leq 5$, $b_6=(1, 1, -1, -1, 0)$, $b_7=(1, -1, 0, 1, 0)$, $b_8=(-1, 1, -1, 0, 1)$ and $b_9=(1, 1, 0, -1, -1)$. Then we see the following:

Lemma 5.2. *The subsemigroup S of $\mathbf{Z}^5$ is saturated.*

Proof. Take $p=(p_1, \cdots, p_5)\in\sum_{i=1}^{9}\mathbf{R}_+b_i\cap\mathbf{Z}^5$. Then it suffices to show that $p\in S$. Moreover, we may assume that $p=\sum_{i=1}^{9} m_ib_i$ with $0\leq m_i<1$, all i. Then we obtain

$$p_1=m_1+m_6+m_7-m_8+m_9\geq 0,$$
$$p_2=m_2+m_6-m_7+m_8+m_9\geq 0,\ p_3=m_3-m_6-m_8\geq -1,$$
$$p_4=m_4-m_6+m_7-m_9\geq -1 \text{ and } p_5=m_5+m_8-m_9\geq 0.$$

To prove that S is saturated it suffices to show that $p\in S$ if $(p_3, p_4)=(-1, -1)$ or $(-1, 0)$ or $(-1, 1)$ or $(0, -1)$. Let $(p_3, p_4)=(-1, -1)$ or $(0, -1)$. We have $m_4+m_7+1=m_6+m_9$, which implies that $p_1\geq 1$ and $p_2\geq 1$. Hence we may assume that

$$p=(1, 1, -1, -1, 0)=b_6\in S.$$

Let $(p_3, p_4)=(-1, 0)$ or $(-1, 1)$. We have $m_3+1=m_6+m_8$, which implies that $m_6>0$ and $p_2\geq 1$. Moreover, we have

$$p_1+p_5=m_1+m_5+m_6+m_7\geq 1,$$

which implies that $p_1\geq 1$ and $p_5\geq 1$. Hence we let $p_1\geq 1$ (resp. $p_5\geq 1$). Then we may assume that

$p=(1, 1, -1, 0, 0)=b_6+b_4\in S$
(resp. $p=(0, 1, -1, 0, 1)=b_8+b_1\in S$). *Q.E.D.*

Now we have the following relations:

$$c_3'a_3+c_4'a_4=c_{10}a_0+(c_{02}+c_{12})a_2,\ c_{30}a_0+c_4'a_4=c_{02}a_2+c_3'a_3,$$
$$c_{01}a_1+c_4'a_4=(c_{10}+c_0')a_0+c_3'a_3,\ c_{12}a_2+c_4'a_4=c_0'a_0+c_{21}a_1+c_3'a_3,$$
$$(c_{10}+c_{30}+c_0')a_0=c_{01}a_1+c_{02}a_2,\ (c_{01}+c_{21})a_1=c_{10}a_0+c_{12}a_2,$$
$$(c_{02}+c_{12})a_2=(c_{30}+c_0')a_0+c_{21}a_1,\ 2c_3'a_3=c_{30}a_0+(c_{01}+c_{21})a_1,$$
$$2c_4'a_4=(c_{10}+c_0')a_0+c_{21}a_1+c_{02}a_2,$$

where all c_i''s and c_{ji}'s are equal to 1. Hence we set

$$g_1=X_3^{c_3'},\ g_2=X_4^{c_4'},\ g_3=X_0^{c_{10}},\ g_4=X_2^{c_{02}},\ g_5=X_1^{c_{01}},$$
$$g_6=X_2^{c_{12}},\ g_7=X_0^{c_{30}},\ g_8=X_0^{c_0'} \text{ and } g_9=X_1^{c_{21}}.$$

Let

$$\pi:\ \mathbf{C}[Y]=\mathbf{C}[Y_1, \cdots, Y_9]\longrightarrow\mathbf{C}[S]=\mathbf{C}[T^s]_{s\in S}$$
$$(\text{resp. } \eta:\ \mathbf{C}[Y]\longrightarrow\mathbf{C}[X]=\mathbf{C}[X_0,X_1,X_2,X_3,X_4])$$

be the $\mathbf{C}$-algebra homomorphism defined by $\pi(Y_i)=T^{b_i}$ (resp. $\eta(Y_i)=g_i$).

Lemma 5.3. *The ideal I_L is generated by the elements of the set $\eta(\mathrm{Ker}\ \pi)$.*

Proof. Let

$$\zeta:\ \mathbf{C}[\mathbf{N}^5]=\mathbf{C}[t_1, t_2, t_3, t_4, t_5]\longrightarrow\mathbf{C}[L]$$

be the $\mathbf{C}$-algebra homomorphism defined by $\zeta(t_i)=t^{w(g_i)}$. Then ζ extends to $\zeta':\ \mathbf{C}[S]\longrightarrow\mathbf{C}[L]$, because we have

$$w(g_1g_2g_3^{-1}g_4^{-1})=c_3'a_3+c_4'a_4-c_{10}a_0-c_{02}a_2=c_{12}a_2=w(g_6),$$
$$w(g_1g_2^{-1}g_4)=c_3'a_3-c_4'a_4+c_{02}a_2=c_{30}a_0=w(g_7),$$
$$w(g_1^{-1}g_2g_3^{-1}g_5)=-c_3'a_3+c_4'a_4-c_{10}a_0+c_{01}a_1=c_0'a_0=w(g_8),$$
$$w(g_1g_2g_4^{-1}g_5^{-1})=c_3'a_3+c_4'a_4-c_{02}a_2-c_{01}a_1$$
$$=c_{10}a_0+c_{12}a_2-c_{01}a_1=c_{21}a_1=w(g_9).$$

Moreover, by the above we obtain $\varphi_L\circ\eta=\zeta'\circ\pi$, which implies that $\eta(\mathrm{Ker}\ \pi)\subseteq\mathrm{Ker}\ \varphi_L=I_L$. Since we have

$$b_1+b_2=b_3+b_4+b_6,\ b_7+b_2=b_4+b_1,\ b_5+b_2=b_3+b_8+b_1,$$
$$b_6+b_2=b_8+b_9+b_1,\ b_3+b_7+b_8=b_5+b_4,\ b_5+b_9=b_3+b_6,$$
$$b_4+b_6=b_7+b_8+b_9,\ 2b_1=b_7+b_5+b_9 \text{ and } 2b_2=b_3+b_8+b_9+b_4,$$

Ker π contains

$$Y_1Y_2-Y_3Y_4Y_6,\ Y_7Y_2-Y_4Y_1,\ Y_5Y_2-Y_3Y_8Y_1,$$
$$Y_6Y_2-Y_8Y_9Y_1,\ Y_3Y_7Y_8-Y_5Y_4,\ Y_5Y_9-Y_3Y_6,$$
$$Y_4Y_6-Y_7Y_8Y_9,\ Y_1^2-Y_7Y_5Y_9 \text{ and } Y_2^2-Y_3Y_8Y_9Y_4,$$

which implies that $\eta(\mathrm{Ker}\ \pi)$ contains the generators for I_L because of Lemma 5.1. Hence we obtain our desired result. *Q.E.D.*

Theorem 5.4. *The gap sequence* $L=\{1, 2, 3, 4, 5, 7, 9, 13\}$ *is Weierstrass.*

Proof. By Lemmas 5.2 and 5.3, L is Weierstrass[3]. *Q.E.D.*

By §1, 2 and Theorems 3.1, 4.4, 5.4, we get the following:

MAIN THEOREM. *Every non-primitive gap sequence of genus 8 except the four sequences* $\{1, 2, 3, 4, 5, 7, 10, 13\}$, $\{1, 2, 3, 4, 5, 7, 9, 15\}$, $\{1, 2, 3, 4, 5, 7, 8, 13\}$ *and* $\{1, 2, 3, 4, 5, 6, 8, 15\}$ *is Weierstrass.*

I do not know whether the above four gap sequences are Weierstrass.

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