

# Non-critical String Theory as Quantum Gravity

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## Abstract

The non-critical Polyakov string was quantized by the KPZ model and the David-Distler-Kawai model. It has been shown that the partition function of the two-dimensional string theory coupled to the quantum gravity can be calculated exactly. Moreover it has been recently obtained that a geodesic distance defined by the transfer matrix formalism for the two-dimensional pure gravity can be regarded as the time variable. Consequently the Hamiltonian operator of the string field theory for central charge,  $c=0$ , is constructed. The string amplitudes can be derived by applying this string field Hamiltonian.

## 1. Introduction

The quantum gravity in four dimensions is one of the most serious, unsolved problems in the recent particle physics. We have never had a completely prospective strategy to this area. In two dimensions, however, we have arrived at the view that the string field actually reduced to the matter fields coupled to the two-dimensional (Liouville) gravity[1][2].

In modern string theory, transition amplitudes between physical strings are described and calculated in terms of conformal field theories (CFT) on the two-dimensional (2D) worldsheets. The matter fields in physically interesting string theories are related to space-time coordinates, describing how the worldsheet embedded in a target space.

At present, the critical string theories are candidates of quantum gravity. The 2D quantum gravity coupled to  $c \leq 1$  matter is equivalent to the  $c \leq 1$  non-critical string theory. The critical bosonic string theory is equivalent to the  $c=25$  non-critical string theory. Therefore the  $c \leq 1$  non-critical string theory has been investigated as the toy models of both the 4D quantum gravity and the critical bosonic string theory.

In the present work, we review the recent progress of the non-critical string field theory. This procedure starts a new formalism called the transfer matrix, which includes an interpretation of a geodesic distance as the time variable[3]. Consequently the Hamiltonian operator of the string field theory for the  $c=0$  pure gravity is constructed. The string amplitudes can be derived by applying this string field Hamiltonian. The  $c=0$  pure gravity is investigated in a new type of gauge called the temporal gauge. We can know that the system is reduced to the quantum mechanics of the loop length  $l$ ,  $c \leq 1$ [4][5].

The organization of this paper is as follows. In the following section 2, we dedicate a brief review of the main results from the 2D quantum gravity in order to understand an important relation between the non-critical string field theory and the 2D quantum gravity.

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In section 3, the transfer matrix formalism is shown and we here discuss that the geodesic distance plays the role of time. In section 4, we consider the case of the  $c=0$  string field theory for the pure gravity. The string field Hamiltonian is constructed and the corresponding Schwinger-Dyson equation is derived.

## 2. 2D Quantum Gravity

Recently the 2D quantum gravity has been investigated from both view points of the Liouville theory[1][2] and the dynamical triangulation (DT)[6][7]. In the Liouville theory (the continuum theory), the path integration of metric is performed. The partition function for the continuum approach is

$$Z = \sum_{h=0}^{\infty} g_c^{2h-2} \int \frac{Dg}{Vol(Diff)} e^{-t \int d^2x \sqrt{g}} \quad (1)$$

$$= \sum_{h=0}^{\infty} g_c^{2h-2} \int \frac{Dg}{Vol(Diff)} \delta\left(\int d^2x \sqrt{g} - A\right), \quad (2)$$

where  $g_c$  is a string coupling constant and  $h$  is genus. These formulars show the case of the pure gravity ( $c=0$ ). In case of (2) Laplace transformation shows up when  $t$  is constant, where  $A$  is the total area of the surface on the 2D manifold.

In the continuum framework we also have the 2D quantum  $R^2$  gravity[8]. As is well-known, surfaces have many spikes corresponding to large local fluctuations. The large scalar curvature  $R$  is localized at the ends of spikes. We disperse the localized curvature all over the surface in order to smooth out surfaces. By adding the  $R^2$  term to the action we suppress wild fluctuations of  $R$ . The  $R^2$  gravity is investigated within the framework of the CFT. As a result, it is shown that the partition function for small area  $A$  is highly suppressed by an exponential factor,

$$e^{-\frac{2\pi(1-h)^2}{m^2 A}}, \quad (3)$$

where  $1/m^2$  is the coefficient of  $R^2$  and  $h$  is the number of handles of the closed orientable surface. We also have the formulation of the  $2+\varepsilon$  quantum gravity[9] in the continuum theory. This renormalizable model is given by generalizing the non-linear sigma model which is regarded as the approach to the string theory. We find that the theory possesses the ultraviolet stable fixed point if the central charge of the matter sector is in the range  $0 < c < 25$ .

On the other hand, we have two kinds of the discrete approaches, which mean the DT approach[6][7] and the matrix model approach[10]. In the DT approach all possible triangulated surfaces are summed up where each triangle is a regular triangle with the same size. The partition function for the discrete version is

$$Z = \sum_{h=0}^{\infty} g_0^{2h-2} \sum_{N_{DT}} \lambda_0^{N_d}, \quad \lambda_0 = e^{-t_0}, \quad (4)$$

where  $N_{DT}$  is the number of the dynamical triangulation on the genus  $h$  surface and  $N_d$  is the

number of triangles. As a result, the correlation function in the continuum approach (the Liouville theory) coincides with that in the DT approach when we obtain the continuum limit. We explore the second discretized version, which is a hermitian  $N \times N$  matrix called a matrix model. The advantage of the matrix model is that it has the exact analytic solution to the problem for  $c \leq 1$  in the large  $N$  limit. The Hermitian 1-matrix model is defined by the partition function,

$$Z = \int D\phi e^{-\frac{N}{\Lambda} \text{tr} V(\phi)}, \quad (5)$$

and

$$D\phi = \prod_i D\phi_{ii} \prod_{i < j} D(R\phi_{ij}) D(T\phi_{ij}), \quad (6)$$

where

$$V(\phi) = \sum_k g_k \phi^k, \quad \phi^\dagger = \phi. \quad (7)$$

We can generate different critical regimes by tuning the coupling constants  $g_k$ . The potentials,  $V(\phi)$ , found by Kazakov[11] serve this property. In the  $m$ -th critical regime the continuum limit of this model is defined[12] by a double scaling limit, where the matrix size gets large,  $N \rightarrow \infty$ , and the constant  $\Lambda$  approaches a critical value,  $\Lambda \rightarrow \Lambda_c$ , while the combination,  $N(\Lambda_c - \Lambda)^{1+1/2m}$ , is kept constant. In general the matrix model provides the most powerful technique for investigating the non-critical string theory.

Furthermore it has been recently obtained that a geodesic distance defined by the transfer matrix formalism[3] for the 2D pure gravity can be regarded as the time variable. In the next section we consider this new procedure to the 2D quantum gravity in detail.

### 3. Transfer Matrix

In this section we present a new formalism[3][13] for the 2D pure gravity ( $c=0$ ), which is called a transfer matrix given by Kawai, Kawamoto, Mogami and Watabiki (KKMW).

Now we consider a cylinder with an entrance loop  $c$  and an exit loop  $c'$ . We defined  $d$  as a geodesic distance from the entrance loop  $c$  to the exit loop  $c'$ . The introduced quantity,  $N(l, l'; d; A)$ , is shown by the following formula,

$$\begin{aligned} N(l, l'; d; A) = & \int \frac{Dg}{\text{Vol}(\text{Diff})} \delta\left(\int d^2x \sqrt{g} - A\right) \\ & \times \delta\left(\int_c \sqrt{g_{\mu\nu}} dx^\mu dx^\nu - l\right) \delta\left(\int_{c'} \sqrt{g_{\mu\nu}} dx^\mu dx^\nu - l'\right) \\ & \times \prod_{p \in c} \delta(d(p, c) - d), \end{aligned} \quad (8)$$

where  $A$  is the total area of the surface on the 2D manifold and  $l$  and  $l'$  are the loop lengths of  $c$  and  $c'$ , respectively. We define  $d(p, c)$  as the minimum distance between the point  $p$  and the point on the loop  $l$ . Moreover the composition law is provided by the following formula,

$$N(l, l'; d; A) = \int_0^\infty dA' \int_0^\infty dl'' \times N(l, l''; d'; A') N(l'', l'; d-d'; A-A'). \quad (9)$$

We define the transfer matrix,  $N(l, l'; d)$ ,

$$N(l, l'; d) = \int_0^\infty dA N(l, l'; d; A) e^{-tA}, \quad (10)$$

where  $t$  is the cosmological constant. A cylinder with the height  $d$  is decomposed into two cylinders with the height  $d'$  and  $d-d'$ .  $N(l, l'; d)$  is regarded as the matrix element for a Hamiltonian,  $H$ ,

$$N(l, l'; d) = \langle l | \exp(-dH) | l' \rangle. \quad (11)$$

Therefore  $N(l, l'; d)$  is also called the proper time evolution kernel. Consequently we obtain the Hamiltonian formalism in which the geodesic distance  $d$  plays the role of the time variable.

A constructive definition of  $N(l, l'; d; A)$  can be given in terms of the DT[3]. By using the  $d$ -step deformation of a loop, the lattice counterpart  $N_{DT}(l, l'; d; n)$  is defined as the number of possible triangulations of a cylinder, where  $n$  is the number of the triangles and  $l, l'$  are links of the loop. The exit loop is one of the connected components obtained after a  $d$ -step deformation of the entrance loop. Consequently the role of the transfer matrix in the DT method,  $N_{DT}(l, l'; d)$ , is obtained. In the continuum limit of the DT the loop length distribution  $f(l, r)$  is obtained by counting the number of loops with the length  $l$  which make the boundaries of area covered within  $r$  steps from a point KKMW predicts the distribution as a function of the scaling variable  $x = l/r^2$  as follows,

$$f(x) \sim \frac{1}{r^2} \left( x^{-5/2} + \frac{1}{2} x^{-3/2} + \frac{14}{3} x^{1/2} \right) e^{-x}. \quad (12)$$

Here we need to refer to a numerical result of the Monte Carlo simulation of the 2D random surface given by the DT under influence of higher order curvature terms. According to the reports by Tsuda and Yukawa[7], the results of the numerical simulation are relatively coincident with the KKMW theoretical prediction. It is shown that the Monte Carlo simulation by the DT method reproduce surprisingly well fractal nature of the surface as predicted by the Liouville theory[1][2] and the string field theory[3][4]. The transfer matrix formalism for the pure gravity is effective to analyze the fractal structure of the quantized surface. Nevertheless we don't go into the procedure of the DT in the 2D quantum gravity in detail more than the above since our purpose is the construction of the string field theory in the continuum approach. The transfer matrix formalism plays an essential role in the construction of the non-critical string theory owing to pushing the transfer matrix formalism forward.

#### 4. $c=0$ String Field Theory

In this section we construct the string field theory ( $c=0$ ) corresponding to the pure gravity. The string field theory is historically considered to be the most promising formalism for perturbative definition of the string theory[14].

First, we construct the string field Hamiltonian operator corresponding to the coordinate in this time. The Hamiltonian includes a three-string vertex and a tadpole. The Schwinger-Dyson equation (or the Wheeler-deWitt equation) for this string field theory is obtained [4].

Let us now consider the evolution of the boundary loop and two kinds of the following processes,

1. The string splits into several strings.
2. The string disappears.

We define  $\Psi^+(l)$  and  $\Psi(l)$  as the creation and annihilation operator of a string (universe) with length  $l(>0)$ , respectively. The following commutation relation is satisfied,

$$[\Psi^+(l), \Psi(l')] = \delta(l - l'), \quad (13)$$

$$\langle 0 | \Psi^+(l) = \Psi(l) | 0 \rangle. \quad (14)$$

$\Psi^+(l)(\Psi(l'))$  creates (annihilates) a string with length  $l$ . These field operators act on the Hilbert space generated from the vacuum. After the time evolution by the Hamiltonian the incident string disappears. The corresponding disk partition function is indicated by the following formula,

$$w(l) = \lim_{d \rightarrow \infty} \langle 0 | \exp(-dH_d) \Psi^+(l) | 0 \rangle, \quad (15)$$

where  $d$  is the Euclidean proper time. The philosophy for this string field theory is relating to the third quantization of the universe. Therefore we regard the above partition function as the Wheeler-deWitt wave function[15]. In terms of the path integral the disk partition function can be also shown as follows,

$$w(l) = \int \frac{Dg}{Vol(Diff)} e^{-\int d^2x \sqrt{g}} \times \delta\left(\int_c \sqrt{g_{\mu\nu}} dx^\mu dx^\nu - l\right). \quad (16)$$

Now we derive the Schwinger-Dyson equation (or the Wheeler-deWitt equation). We first consider the existence of the large  $d$  limit in the disk partition function  $w(l)$ , which implies the condition,

$$\lim_{d \rightarrow \infty} \frac{\partial}{\partial d} \langle 0 | \exp(-dH_d) \Psi^+(l) | 0 \rangle = 0, \quad (17)$$

where the above equation is corresponding to the Wheeler-deWitt equation for the wave function  $w(l)$ . We find that the partition function,

$$\langle 0 | \exp(-dH_d) \Psi^+(l) | 0 \rangle, \quad (18)$$

does not evolve in the large limit  $d$ , the time evolution. The string field Hamiltonian describes evolutions of a string in the coordinate frame defined by the KKMW method, the transfer matrix formalism. We fix the form of the Hamiltonian  $H_d$ ,

$$\begin{aligned} H_d = & \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1) \Psi^\dagger(l_2) \Psi(l_1 + l_2) (l_1 + l_2) \\ & + \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi^\dagger(l_1) K(l_1, l_2) \Psi(l_2) \\ & + \int_0^\infty dl \rho(l) \Psi(l), \end{aligned} \quad (19)$$

where each term in the above formula describes the splitting of the loop for the first term, the kinetic term representing the amplitude of cylinder with an initial and a final loop length,  $l_1$  and  $l_2$  for the second term and the tadpole term representing the cap amplitude for instantaneous vanishing of the loop with the length  $l$ . Therefore after using  $H_d |0\rangle = 0$ , the following relation is obtained from the equation (17),

$$\lim_{d \rightarrow \infty} \frac{\partial}{\partial d} \langle 0 | \exp(-dH_d) [H_d, \Psi^\dagger(l)] | 0 \rangle = 0. \quad (20)$$

By using the following factorization,

$$\begin{aligned} \lim_{d \rightarrow \infty} \langle 0 | \exp(-dH_d) \Psi^\dagger(l_1) \Psi^\dagger(l_2) | 0 \rangle &= \lim_{d \rightarrow \infty} \langle 0 | \exp(-dH_d) \Psi^\dagger(l_1) | 0 \rangle \\ &\quad \times \lim_{d \rightarrow \infty} \langle 0 | \exp(-dH_d) \Psi^\dagger(l_2) | 0 \rangle, \end{aligned} \quad (21)$$

we can construct the Schwinger-Dyson equation (or the Wheeler-deWitt equation) as follows,

$$l \int_0^l dl_1 w(l - l_1) + \rho(l) = 0. \quad (22)$$

Let us now consider the orientable 2D manifold (mfd)  $M$  with  $k$  boundaries,  $c_1, c_2, \dots, c_k$ . First we put the metric  $g_{\mu\nu}$  on the 2D mfd  $M$ . Next we provide topological slice cuts to the 2D mfd  $M$  with genus  $h$  as well as we treat the height,  $c_1 \cup c_2 \cup \dots \cup c_k$ , by the Morse theory in math. We can define the geodesic distance for the height  $p \in M$ ,

$$\begin{aligned} d(p) &= d(p; c_1 \cup c_2 \cup \dots \cup c_k) \\ &= \min d(p, q), \quad q \in c_1 \cup c_2 \cup \dots \cup c_k. \end{aligned} \quad (23)$$

In this case the creation operators generate the following state,

$$\Psi^\dagger(l_1) \dots \Psi^\dagger(l_k) | 0 \rangle. \quad (24)$$

The above formula means a certain state that  $k$  spaces (universes) with the loop length  $l_1, l_2, \dots, l_k$  exist. Furthermore the corresponding partition function (the Wheeler-deWitt wave function) for the surfaces with  $k$  boundaries should be shown as

$$\begin{aligned} w(l_1) \dots w(l_k) &= \lim_{d \rightarrow \infty} \langle 0 | \exp(-dH) \Psi^\dagger(l_1) \dots \Psi^\dagger(l_k) | 0 \rangle \\ &= \sum_{h=0}^{\infty} g_c^{h+k-1} \int \frac{Dg}{\text{Vol}(\text{Diff})} e^{-\int d^2x \sqrt{g}} \\ &\quad \times \prod_{j=1}^k \delta\left(\int_{c_j} \sqrt{g_{\mu\nu}} dx^\mu dx^\nu - l_j\right), \end{aligned} \quad (25)$$

where the boundary  $c_j$  is corresponding to the loop length  $l_j$  and  $g_c$  is the string coupling constant. Because of the merging process, this expression includes the contributions from the surface with  $h$  handles. The contribution from the connected surfaces with  $h$  handles and  $k$  boundaries is proportional to  $g_c^{h+k-1}$ . A  $k$ -string partition function corresponds to the worldsheets with  $k$  boundaries, each of which describes an external string state. The string amplitude can be obtained by solving the string field Schwinger-Dyson equation (the Wheeler-deWitt equation) as follows,

$$\lim_{d \rightarrow \infty} \partial_a \langle 0 | \exp(-dH) \Psi^*(l_1) \cdots \Psi^*(l_k) | 0 \rangle = 0. \quad (26)$$

This equation means that the string amplitudes do not change if we consider the time evolution operator on all the external string states.

## 5. Discussion

In the previous section the string field Hamiltonian for the  $c=0$  string theory was constructed. We can consider the string field Hamiltonian for the  $c=1/2$  string[4]. The  $c=1/2$  matter theory is actually represented by the continuum limit of the Ising model. In this case we need construct the string field theory with the up or down spins on the string. Moreover recently the string field Hamiltonian for  $c=1-6/m(m+1)$  string theory in the temporal gauge[5] is constructed. Consequently the  $W$  constraints are deduced from the string field Schwinger-Dyson equation.

There is some possibility that the non-critical string field theory may be constructed on the twistor manifold[16]. Accordingly, it means that the 2D quantum gravity may be relating to the non-critical twistor theory improved from the Penrose's method[17]. In our opinion, namely, what is the existence corresponding to the CFT in the twistor manifold? In this case the twistor manifold is not the target space including the structure of the space-time, in the string theory, but the target space and the 2D world-sheet should be translated into the twistor language. It shows that the non-critical twistor creates and annihilates the 2D space-time. A detail inquiry into this problem will be given in the next papers[18].

The quantum gravity should be naturally regarded as the post target area in the particle physics theory. The quantum gravity has been thought of showing its true character gradually owing to the recent progress in several years, as described in the sections above.

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