The Stability Sufficient Condition for N-Dimensional Recursive Digital Filters

Sadayoshi TAKAHASHI*, Hisayuki TATSUMI*
and Tomoyuki ARAKI*

Abstract

For 1-D and 2-D recursive digital filters, the stability necessary and sufficient conditions were obtained [1]. In this paper, for 3 or higher dimensional recursive filters, the stable sufficient conditions are presented. Especially, for 3-D filters, the necessary and sufficient conditions are mathematically obtained. And also, for the multilinear 4-variate example, the necessary and sufficient conditions are obtained.

1. Introduction

On N-dimensional recursive digital filters, we have proposed that the stable necessary and sufficient condition for two dimensional digital filters is represented in polynomials composed of coefficients of the denominators of their transfer functions[1]. This paper describes that we can obtain the stable condition represented in literal coefficients as sufficient condition for 3-or higher-dimensional digital filters.

2. Some Theories

It have been shown in Ref. (2) that the stable condition for N-dimensional digital filters is reduced to the positive value problem of polynomials for (N-1) variables. Accordingly, the stable condition for 2-dimensional digital filters can be solved perfectly by applying Sturm's theorem, etc[3]. and Takahashi's lemma[1] as the positive value problem of polynomials for one variable.

3. The Positive Value Problem for Multivariable Polynomials

As to multivaliable polynomials, the stability condition is representable as the sum of product form of resultants by aiming at one valiable, considering the other variables to be parameters and applying the theory in Sec. 2.

$$Y=\sum_{i=1}^{n} X_{i,1}\cdot X_{i,2}\cdot\cdots\cdot X_{i,j_i} \tag{1}$$

(where j_i is an integer number for any i)

Received, 1994. 9. 20.
* 情報工学科

And the condition satisfying $Y=1$ is

$$X_{i,1}\cdot X_{i,2}\cdot\cdots\cdot X_{i,j_i}=1 (i=1 \text{ or } 2 \text{ or } \cdots \text{ or } n). \tag{2}$$

Therefore, the sufficient condition satisfying $Y=1$ is that $X_{i,1}\cdot X_{i,2}\cdot\cdots\cdot X_{i,j_i}=1$ holds for each prime implicant, that is, $X_{i,1}>0$ $(j=1, \cdots, j_i)$ holds, $X_{i,1}$ forms a polynomial that is deleted one valiable from the beginning polynomial. Consequently, we can obtain the conditional formula represented in only literal coefficients that do not include variables by applying the above algorithm recursively.

4. Examples

Let the denominator polynomial of a transfer function to be

$$B(Z_1, Z_2, Z_3, Z_4)=1+aZ_1Z_2Z_3+bZ_2Z_3Z_4+cZ_3Z_4Z_1 \\ +dZ_4Z_1Z_2+eZ_1Z_2Z_3Z_4.$$

The sufficient condition that $B(Z_1, Z_2, Z_3, Z_4)$ can be stable are given by the following steps.

(i) Substitute Z_i for $Z_i=(1-S_i)/(1+S_i)$ $(i=1, 2, 3, 4)$, where $S_1=ix$, $S_2=jy$, $S_3=jz$ and $S_4=jw$. Then rearrange it in the real part and the imaginary part.

$$\begin{aligned} B_A(x, y, z, w)=&z\{A_1wxy+A_2w+A_3x+A_4y\}+B_1+B_2xy+B_3wx+B_4wy \\ &+j[z\{C_1+C_2xy+C_3wx+C_4wy\}+D_1wxy+D_2w+D_3x+D_4y] \end{aligned} \tag{3}$$

(where the coefficient A_i, B_i, C_i are polynomials composed of a, b, c, d, e).

(ii) Applying the theories described in sec. 2 to $B_A(x, y, z, w)$, we get

$$R=\begin{vmatrix} A_1wxy+A_2w+A_3x+A_4y & B_1+B_2xy+B_3wx+B_4wy \\ C_1+C_2xy+C_3wy+C_4wy & D_1wxy+D_2w+D_3x+D_4y \end{vmatrix}<0. \tag{4}$$

(for any real x, y, z)

By arranging the left side hand in the above inequality for x, we get

$$\begin{aligned} R=&x^2(\alpha_1w^2y^2+\beta_1w^2+\gamma_1y^2+\delta_1wy+\varepsilon_1)+x(\alpha_2w^2y+\beta_2wy^2+\gamma_2w+\delta_2y) \\ &+\alpha_3w^2y^2+\beta_3wy+\gamma_3 \end{aligned} \tag{5}$$

Since R is a quadratic equation, the following conditions are obtained without considering the sequence of resultants. Then the discriminant is negative.So,

$$\begin{aligned} &\{\alpha_2w^2y+\beta_2wy^2+\gamma_2w+\delta_2y\}^2 \\ &-4(\alpha_1w^2y^2+\beta_1w^2+\gamma_1y^2+\delta_1wy+\varepsilon_1)(\alpha_3w^2y^2+\beta_3wy+\gamma_3)<0 \end{aligned} \tag{6}$$

(where the coefficients α_i, β_i, γ_i δ_i $(i=1\sim3)$ are polynomials with a, b, c, d, e)

By arranging the above inequality for y, we get

$$
\begin{aligned}
f(y)=&\,w^2y^4\{Aw^2+B\}\\
&+y^3\{2\beta_2 w(\alpha_2 w^2+\delta_2)-4\beta_3 w(\alpha_1 w^2+\gamma_1)-4\delta_1 w^3\alpha_3\}\\
&+wy^3\{Cw^2+D\}+y_2\{Ew^4+Fw^2+G\}\\
&+yw\{Hw^2+I\}+Jw^2+K\\
&>0
\end{aligned}
\tag{7}
$$

(where the coefficients A, B, C, D, E, F, G, H, I, J, K are polynomials with a, b, c, d, e.)

As for this polynomial, we try to obtain the sequence of resultants ($R_0=1$, R_1, R_2, R_3, R_4) from $f(y)$ an $\dfrac{d}{dy}f(y)$.

The following inequalities satisfy the condition where $f(y)$ is positive by the result of Ref. 4.

$$R_4>0 \text{ and } \bar{R}_2+\bar{R}_3>0. \tag{8}$$

Consequently, the sufficient condition can be obtained as follows.

$$(1)\quad R_4>0,\ R_2<0, \tag{9}$$

or

$$(2)\quad R_4>0,\ R_3<0, \tag{10}$$

(iii) R_1, R_2, R_3, R_4 are calculated as follows.

$$R_1=4w^4\{Aw^2+B\}^2>0 \tag{11}$$

$$
\begin{aligned}
R_2=&\begin{vmatrix} 4w^2(Aw^2+B) & 3w(Cw^2+D) & 2(Ew^4+Fw^2+G)\\ w^2(Aw^2+B) & w(Cw^2+D) & Ew^4+Fw^2+G\\ 0 & 4w^2(Aw^2+B) & 3w(Cw^2+D)\end{vmatrix}\\
&\times(Aw^2+B)w^2\\
&=w^6(Aw^2+B)^2\times(\text{the 3 order equation for } w^2)
\end{aligned}
\tag{12}
$$

$$R_3=(\text{at most, the 12 order equation for } w^2) \tag{13}$$

$$R_4=(\text{at most, the 32 order equation for } w) \tag{14}$$

(See an appendix for R_3 and R_4.)

(iv) As to R_2, R_3 and R_4, for variable w, their resultants are obtatined as expressions composed of literal coefficients. Then, the condition corresponding to the sufficient condition in (ii) can be obtained by investigating the number of sign changes in their resultants.

(v) In practice, we calculate these resultants.

There exist resultants of R_2^0, R_2^1, ..., R_2^{16} for R_2.

$$\sum_{i=1}^{16} Sign\ (R_2^{i-1}\cdot R_2^i)=0 \tag{15}$$

In the same manner, we can derive resultants of R_3^0, R_3^1, ..., R_3^{24} for R_3 and R_4^0, R_4^{32} for R_4, respectively.

$$\sum_{i=1}^{24} Sign(R_3^{i-1}\cdot R_3^i)=0 \tag{16}$$

$$\sum_{i=1}^{32} Sign(R_4^{i-1} \cdot R_4^i) = 0 \tag{17}$$

The formulas (15), (16) and (17) can be expressed as the sum of prime implicants by minimizing them when we consider them to be logic functions.

5. Conclusion

This paper descibed the method to find the sufficient condition of stability for N-dimensional recursive digital filters, especially, as an example for a linear 4-dimensional digital, filter.

As for an interesting point :

(1) The algorithm to establish stable domains as subsets one after another recursively is similar to the method to determine filters in set theory. And for any 3-dimensional digital filter, by extending the logic formula that expresses the filter itself to the sum of product form, this problem can be reduced to the positive value problem of an envelope. In this case, we can get the necessary and sufficient condition in enough technical accuracy.

Acknowledgments

The authors wish to express gratitudes to Professor N.K. Bose of the Pennsylvania State University for his useful discussions.

References

1) Takahashi, S. and Tatsumi, T.: "The stable Condition of Literal Coefficients for Linear 2-D Digital Systems" (in Japanese), Proccedings of the 6th Karuizawa Workshop on Circuits and Systems, pp. 31-36, (Apr. 1993).

2) Takahashi, S. and Tsujii, S.: "Stability Condition for N-Dimensional Digital Filters and Extreme Problem for Polynomials of Several Variables" (in Japanese), Trans. IEICE, Vol. 57-A, No. 10, pp. 746-752, (Oct. 1974).

3) Takagi, T.: "Lectures on Algebra" (in Japanese), Kyoritsu Publ. Co. (1968).

4) Takahashi, S.: "The stable Condition of Literal Coefficients for Linear 2-D Digital Systems" (in Japanese, Proccedings of the 7th Digital Signal Processing Symposium, pp. 362-367, (Nov. 1992)

Appendix

$$
R_3=\begin{vmatrix}
w^2(Aw^2+B) & w(Cw^2+D) & (Ew^4+Fw^2+G) & w(Hw^2+I) \\
0 & 4w^2(Aw^2+B) & 3w(Cw^2+D) & 2(Ew^4+Fw^2+G) \\
0 & w^2(Aw^2+B) & w(Cw^2+D) & Ew^4+Fw^2+G \\
0 & 0 & 4w^2(Aw^2+B) & 3w(Cw^2+D) \\
0 & 0 & w^2(Aw^2+B) & w(Cw^2+D) \\
0 & 0 & 0 & 4w^2(Aw^2+B) \\
 & & Jw^2+K & 0 \\
 & & w(Hw^2+I) & 0 \\
 & & w(Hw^2+I) & Jw^2+K \\
 & & 2(Ew^4+Fw^2+G) & w(Hw^2+I) \\
 & & Ew^4+Fw^2+G & w(Hw^2+I) \\
 & & 3w(Cw^2+D) & 2(Ew^4+Fw^2+G)
\end{vmatrix}
$$

$$
R_4=\begin{vmatrix}
w^2(Aw^2+B) & w(Cw^2+D) & (Ew^4+Fw^2+G) & w(Hw^2+1) \\
0 & 4w^2(Aw^2+B) & 3w(Cw^2+D) & 2(Ew^4+Fw^2+G) \\
0 & w^2(Aw^2+B) & w(Cw^2+D) & Ew^4+Fw^2+G \\
0 & 0 & 4w^2(Aw^2+B) & 3w(Cw^2+D) \\
0 & 0 & w^2(Aw^2+B) & w(Cw^2+D) \\
0 & 0 & 0 & 4w^2(Aw^2+B) \\
0 & 0 & 0 & w^2(Aw^2+B) \\
0 & 0 & 0 & 0 \\
Jw^2+K & 0 & 0 & 0 \\
w(Hw^2+I) & 0 & 0 & 0 \\
w(Hw^2+I) & Jw^2+K & 0 & 0 \\
2(Ew^4+Fw^2+G) & w(Hw^2+I) & 0 & 0 \\
Ew^4+Fw^2+G & w(Hw^2+I) & Jw^2+K & 0 \\
3w(Cw^2+D) & 2(Ew^4+Fw^2+G) & w(Hw^2+1) & 0 \\
w(Cw^2+D) & Ew^4+Fw^2+G & w(Hw^2+1) & Jw^2+K \\
4w^2(Aw^2+B) & 3w(Cw^2+D) & 2(Ew^4+Fw^2+G) & w(Hw^2+I)
\end{vmatrix}
$$