

Minimal Primitive Schubert Indices

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Abstract

Let $a_0 \leq a_1 \leq \dots \leq a_{g-1}$ be g non-negative integers which are less than or equal to $g-1$. Then $\alpha = (a_0, a_1, \dots, a_{g-1})$ is said to be a *Schubert index*. The integer g is called the *genus* of α , which is denoted by $g(\alpha)$. We set $a(\alpha) = \min \{i \mid a_i \neq 0\} + 1$ and $\gamma(\alpha) = a_{g-1} + g$. If $2a(\alpha) > \gamma(\alpha)$, then α is said to be *primitive*. In this paper we define a partial order on the set of primitive Schubert indices useful for investigating geometric properties of these indices. We will give a necessary and sufficient condition on a primitive Schubert index α such that α is minimal. Using this result the minimal primitive Schubert indices $\alpha = (a_0, a_1, \dots, a_{g-1})$ with $\sum_{i=0}^{g-1} a_i \leq g+1$ are completely determined. Moreover, we investigate the Schubert indices α of genus g with $a(\alpha) = g-1$ satisfying the semigroup condition.

§1. Minimal primitiveness

In the first section we define a partial order on the set of primitive Schubert indices. We will give a description of the minimal indices by this order.

Let $\alpha = (a_0, a_1, \dots, a_{g-2})$ and $\beta = (\beta_0, \beta_1, \dots, \beta_{g-1})$ be primitive Schubert indices of genus $g-1$ and g respectively. Then we define $\alpha \implies \beta$ if one of the following holds:

- (1) $\beta_0 = 0, a_i = \beta_{i+1}$ for any i with $0 \leq i \leq g-2$.
- (2) $\beta_0 = 0$ and there exists an integer j with $0 \leq j \leq g-2$

such that $a_j = \beta_{j+1} - 1$ and $a_i = \beta_{i+1}$ for $i \geq 0$ with $i \neq j$.

Using the above we define a partial order on the set of primitive Schubert indices as follows: $\alpha \leq \beta$ means that there exists a sequence

$$\alpha = \alpha^{(n)} \implies \alpha^{(n-1)} \implies \dots \implies \alpha^{(1)} \implies \alpha^{(0)} = \beta$$

of primitive Schubert indices. We say that a primitive Schubert index β is *minimal* if there are no primitive Schubert indices α such that $\alpha \leq \beta$ and $\alpha \neq \beta$.

Now we explain a geometric meaning of the partial ordering. For any nonsingular pointed curve (C, P) of genus g , $H(P)$ denotes the set of integers which are pole orders at P of regular functions on $C \setminus \{P\}$. We set $L(P) = N \setminus H(P)$ where N denotes the additive semigroup of non-negative integers. Then it is well-known that the number of the set $L(P)$ is equal to g . Hence we set $L(P) = \{l_1 < l_2 < \dots < l_g\}$. For any Schubert index $\alpha = (a_0, a_1, \dots, a_{g-1})$ of genus g we may define a locally closed subset \mathcal{C}_α of the moduli space $\mathcal{M}_{g,1}$ of nonsingular pointed curves of genus g by $\mathcal{C}_\alpha = \{(C, P) \in \mathcal{M}_{g,1} \mid \alpha(P) = \alpha\}$, where we set $\alpha(P) = (l_1 - 1, l_2 - 2, \dots, l_g - g)$. Then the *weight* $w(\alpha)$ of α , i.e., $w(\alpha) = \sum_{i=0}^{g-1} a_i$, gives an upper bound for the codimension of any component of \mathcal{C}_α in $\mathcal{M}_{g,1}$. A nonsingular pointed curve $x = (C, P) \in \mathcal{C}_\alpha$ is *dimensionally proper* if $\mathcal{C}_\alpha \subset \mathcal{M}_{g,1}$ has codimension $w(\alpha)$ in a neighborhood

of x . We say that α is *dimensionally proper* if \mathcal{C}_α contains a dimensionally proper point $x = (C, P)$. Then Eisenbud-Harris¹⁾ showed that

if $\alpha \leq \beta$ and if α is dimensionally proper, then so is β .

Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g-1})$ be a Schubert index of genus g . Then to state the main theorem in this section we use the notation as follows: We set $\gamma(\alpha) = \alpha_{g-1} + g$.

Theorem 1.1. *Let $\beta = (\beta_0, \beta_1, \dots, \beta_{g-1})$ be a primitive Schubert index of genus g and weight $\geq g-1$ with $\beta \neq (0^{g-1}, g-1)$. Then there exists a sequence*

$$\alpha^{(n)} = (\alpha_0^{(n)}, \alpha_1^{(n)}, \dots, \alpha_{g-1-n}^{(n)}) \implies \alpha^{(n-1)} \implies \dots \implies \alpha^{(1)} \implies \alpha^{(0)} = \beta$$

such that $\alpha^{(i)}$ is a primitive Schubert index of genus $g-i$ and

$$2a(\alpha^{(n)}) = \gamma(\alpha^{(n)}) + 1, \alpha_{g-2-n}^{(n)} = \alpha_{g-1-n}^{(n)}, \alpha_{a(\alpha^{(n)})-1}^{(n)} \geq 2.$$

Proof. First we may assume that $\beta_{g-1} = \beta_{g-2}$. In fact, let $\beta_{g-1} > \beta_{g-2}$. Since β is of weight $\geq g-1$ and is not equal to $(0^{g-1}, g-1)$, we have $\beta_{g-2} \geq 1$. We set

$$\alpha_i = \beta_{i+1} \text{ for any } i=0, 1, \dots, g-3 \text{ and } \alpha_{g-2} = \beta_{g-1} - 1.$$

Then we have

$$2a(\alpha) = 2(a(\beta) - 1) \geq \beta_{g-1} + g - 1 = \alpha_{g-2} + (g-1) + 1,$$

which implies that $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g-2})$ is a primitive Schubert index of genus $g-1$ and weight $\geq (g-1)-1$. Moreover, we have $\alpha \neq (0^{g-2}, g-2)$ because of $\alpha = (0^{g-3}, \beta_{g-2}, \beta_{g-1}-1)$ with $\beta_{g-2} \neq 0$. Using the above method successively we may assume that $\beta_{g-1} = \beta_{g-2}$.

Moreover, we may assume that $\beta_{a(\beta)-1} \geq 2$. In fact, let $\beta_{a(\beta)-1} < 2$. Then $\beta_{a(\beta)-1} = 1$. We set

$$\alpha_i = \beta_{i+1} = 0 \text{ for any } i \text{ with } 0 \leq i \leq a(\beta) - 3,$$

$$\alpha_{a(\beta)-2} = \beta_{a(\beta)-1} - 1 = 0 \text{ and } \alpha_j = \beta_{j+1} \text{ for any } j \text{ with } a(\beta) - 1 \leq j \leq g-2.$$

Then we have

$$2a(\alpha) = 2a(\beta) > \beta_{g-1} + g = \alpha_{g-2} + (g-1) + 1 > \alpha_{g-2} + (g-1),$$

which implies that $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g-2})$ is a primitive Schubert index of genus $g-1$ and weight $\geq (g-1)-1$. Moreover, we have $\alpha \neq (0^{g-2}, g-2)$. In fact, suppose that $\alpha = (0^{g-2}, g-2)$. Then we must have $\beta = (0^{g-2}, 1, g-2)$, which is not primitive. This is a contradiction. Using the above method successively we may assume that $\beta_{a(\beta)-1} \geq 2$. Hence we may assume that

$$\beta = (0^{j_0}, \beta_{j_0}, \dots, \beta_{g-1})$$

where $\beta_{j_0} \geq 2$ and $\beta_{g-2} = \beta_{g-1}$. In this case we have $a(\beta) = j_0 + 1$.

Lastly we will show that there exists a primitive Schubert index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g(a)-1})$ such that

$$\alpha_{a(a)-1} \geq 2, \alpha_{g(a)-2} = \alpha_{g(a)-1}, 2a(\alpha) = \gamma(\alpha) + 1 \text{ and } \alpha \implies \dots \implies \beta.$$

Let $j_0 = g-2$. Then we have $\beta = (0^{g-2}, \beta_{g-2}, \beta_{g-1})$ with $2 \leq \beta_{g-2} = \beta_{g-1} \leq g-3$ because of $2a(\beta)$

$\geq \gamma(\beta) + 1$. If $0 \leq 2m \leq 2g - 2\beta_{g-1} - 6$, then we set

$$\alpha^{(2m)} = (0^{g-2-2m}, \beta_{g-2} - m, \beta_{g-1} - m).$$

We note that

$$\begin{aligned} g - 2 - 2m &\geq g - 2 - (2g - 2\beta_{g-1} - 6) \\ &= 2\beta_{g-1} - g + 4 \geq g - 1 - g + 4 = 3 \end{aligned}$$

and

$$\beta_{g-2} - m \geq \beta_{g-2} - g + \beta_{g-1} + 3 \geq g - 1 - g + 3 = 2.$$

Then we have

$$\begin{aligned} 2\alpha(\alpha^{(2m)}) - 2\gamma(\alpha^{(2m)}) - 1 &= 2(g - 1 - 2m) - (\beta_{g-1} - m + g - 2m) - 1 \\ &= g - m - \beta_{g-1} - 3 \geq g - (g - \beta_{g-1} - 3) - \beta_{g-1} - 3 = 0, \end{aligned}$$

which implies that $\alpha^{(2m)}$ is primitive. If $0 \leq 2m + 1 \leq 2g - 2\beta_{g-1} - 6$, then we set

$$\alpha^{(2m+1)} = (0^{g-2-2m-1}, \beta_{g-2} - m - 1, \beta_{g-1} - m).$$

We note that

$$\begin{aligned} g - 2 - 2m - 1 &\geq g - 2 - (2g - 2\beta_{g-1} - 6) \\ &= 2\beta_{g-1} - g + 4 \geq g - 1 - g + 4 = 3 \end{aligned}$$

and

$$\begin{aligned} 2(\beta_{g-2} - m - 1) &\geq 2\beta_{g-1} - 1 - (2g - 2\beta_{g-1} - 6) \\ &= 4\beta_{g-1} - 2g + 5 \geq 2g - 2 - 2g + 5 = 3. \end{aligned}$$

Then we have

$$\begin{aligned} 2\alpha(\alpha^{(2m+1)}) - 2\gamma(\alpha^{(2m+1)}) - 1 &= 2(g - 2 - 2m) - (\beta_{g-1} - m + g - 2m - 1) - 1 \\ &= g - m - \beta_{g-1} - 3 > g - (g - \beta_{g-1} - 3) - \beta_{g-1} - 3 = 0, \end{aligned}$$

which implies that $\alpha^{(2m+1)}$ is primitive. Then we get a sequence

$$\begin{aligned} \alpha &= \alpha^{(2g-2\beta_{g-1}-6)} = (0^{2\beta_{g-1}-g+4}, 2\beta_{g-1}-g+3, 2\beta_{g-1}-g+3) \\ &\implies \dots \implies \alpha^{(2)} = (0^{g-4}, \beta_{g-2}-1, \beta_{g-1}-1) \implies \alpha^{(1)} \\ &= (0^{g-3}, \beta_{g-2}-1, \beta_{g-1}) \implies \alpha^{(0)} = (0^{g-2}, \beta_{g-2}, \beta_{g-1}) = \beta \end{aligned}$$

of primitive Schubert indices. Then we get

$$2\beta_{g-1} - g + 3 \geq g - 1 - g + 3 = 2.$$

Moreover, we have

$$\begin{aligned} 2\alpha(\alpha) &= 2(2\beta_{g-1} - g + 5) = 2\beta_{g-1} - g + 3 + 2\beta_{g-1} - g + 6 + 1 \\ &= \gamma(\alpha) + 1. \end{aligned}$$

Let $j_0 \leq g - 3$. We set $2\alpha(\beta) = \gamma(\beta) + d$ with $d \geq 1$. If $d = 1$, we set $\alpha = \beta$. Let $d \geq 2$. If $\beta_{j_0} \geq 3$, then we set $\beta' = (\beta'_0, \beta'_1, \dots, \beta'_{g-2})$ where

$$\beta'_i = \beta_{i+1} \text{ for } 0 \leq i \leq g - 2 \text{ with } i \neq j_0 - 1 \text{ and } \beta'_{j_0-1} = \beta_{j_0} - 1 \geq 2.$$

Then we have

$$2a(\beta') = 2j_0 = 2a(\beta) - 2 = \gamma(\beta) + d - 2 = \gamma(\beta') + d - 1.$$

By induction β' satisfies our desired properties, which implies that so does β . If $\beta_{j_0} = 2$, then we set $\beta' = (\beta'_0, \beta'_1, \dots, \beta'_{g-3})$ where

$$\beta'_i = 0 \text{ for } 0 \leq i \leq j_0 - 2 \text{ and } \beta'_i = \beta_{i+2} \text{ for } j_0 - 1 \leq i \leq g - 3.$$

Then we have

$$2a(\beta') = 2j_0 = 2a(\beta) - 2 = \gamma(\beta) + d - 2 = \gamma(\beta') + d.$$

If $\beta_{j_0+1} = \beta'_{j_0-1} \geq 3$, then by the above argument β' satisfies our desired properties, which implies that so does β . Hence the remaining case is the following :

$$\beta_i = 2 \text{ for any } i \text{ with } j_0 \leq i \leq g - 3.$$

Then we have

$$\begin{aligned} (0^{h-2}, a_{h-2}, a_{h-1}) &\leq \beta \text{ where } h = g - 2(g - j_0 - 2) \\ &= 2j_0 + 4 - g. \end{aligned}$$

Since by the above argument $(0^{h-2}, a_{h-2}, a_{h-1})$ satisfies our desired properties, so does β .

Q.E.D.

Corollary 1.2. Let $\beta = (\beta_0, \beta_1, \dots, \beta_{g-1})$ be a primitive Schubert index of genus g and weight $\geq g - 1$ with $\beta \neq (0^{g-1}, g - 1)$. Then the following are equivalent :

- (1) β is minimal.
- (2) $2a(\beta) = \gamma(\beta) + 1$, $\beta_{g-2} = \beta_{g-1}$ and $\beta_{j_0} \geq 2$ where $j_0 = \text{Min} \{j \mid \beta_j \neq 0\}$.

Proof. By Theorem 1.1 (1) implies (2). Let β be a primitive Schubert index satisfying the condition (2). Assume that β is not minimal, i.e., there exists a primitive Schubert index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g-2})$ of genus $g - 1$ such that $\alpha \implies \beta$. First we consider the case where $\alpha_i = \beta_{i+1}$ for any $i = 0, 1, \dots, g - 2$. Since we have

$$\alpha = (\beta_1, \beta_2, \dots, \beta_{g-1}) = (0^{j_0-1}, \beta_{j_0}, \dots, \beta_{g-1}),$$

we get

$$2a(\alpha) = 2j_0 = 2a(\beta) - 2 = \gamma(\beta) - 1 = \alpha_{g-2} + g - 1 = \gamma(\alpha).$$

This is a contradiction. Secondly we consider the case where there exists j with $0 \leq j \leq g - 2$ such that $\alpha_j = \beta_{j+1} - 1$ and $\alpha_i = \beta_{i+1}$ for all $i \neq j$.

Since

$$\beta_{j_0} \geq 2, \beta_{g-2} = \beta_{g-1} \text{ and } \alpha_{g-3} \leq \alpha_{g-2},$$

we have $j \leq g - 3$, which implies that

$$a(\alpha) = j_0 \text{ and } \gamma(\alpha) = \beta_{g-1} + g - 1.$$

Hence we get

$$2a(\alpha) = 2a(\beta) - 2 = \gamma(\beta) - 1 = \beta_{g-1} + g - 1 = \gamma(\alpha),$$

which is a contradiction.

Q.E.D.

§2. Minimal primitive Schubert indices of genus g and weight $\leq g+1$

In the second section we will completely determine minimal primitive Schubert indices of genus g and weight $\leq g+1$.

Proposition 2.1. *If $\beta \neq (0)$ is either a primitive Schubert index of genus g and weight $\leq g-2$ or a Schubert index $(0^{g-1}, g-1)$, then it is not minimal. More explicitly, there exists a sequence*

$$(0) = \alpha^{(0)} \implies \alpha^{(1)} \implies \dots \implies \alpha^{(g-1)} = \beta$$

of primitive Schubert indices.

Proof. Let $\beta \neq (0)$ be a primitive Schubert index of genus g and weight $\leq g-2$. We set $j_0 = \min \{j \mid \beta_j \neq 0\}$. If $j_0 = g-1$, then $\beta = (0^{g-1}, g-i)$ with $2 \leq i \leq g-1$. Then we have $(0^i) \leq \beta$. Let $j_0 \leq g-2$. If $\beta_{j_0} = 1$, then we set $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g-2})$ where $\alpha_{j_0-1} = \beta_{j_0} - 1 = 0$ and $\alpha_i = \beta_{i+1}$ for all $i \neq j_0$. Since

$$2a(\alpha) = 2(j_0 + 1) = 2a(\beta) > \gamma(\beta) = \gamma(\alpha) + 1 > \gamma(\alpha),$$

α is a primitive Schubert index with $\alpha \leq \beta$. Let $\beta_{j_0} \geq 2$. We set $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{g-2})$ where $\alpha_{j_0-1} = \beta_{j_0} - 1 > 0$ and $\alpha_i = \beta_{i+1}$ for all $i \neq j_0$. Then we have

$$a(\alpha) = j_0 = a(\beta) - 1 \text{ and } \gamma(\alpha) = \beta_{g-1} + g - 1 = \gamma(\beta) - 1.$$

Since by the assumption we have

$$g-2 \geq w(\beta) = \beta_{g-1} + \sum_{i=j_0}^{g-2} \beta_i \geq \beta_{g-1} + 2(g-1-j_0),$$

we get

$$2a(\alpha) = 2j_0 \geq \beta_{g-1} + g = \gamma(\alpha) + 1 > \gamma(\alpha),$$

which implies that α is primitive with $\alpha \leq \beta$. Using the above methods successively we get $(0) \leq \beta$. If $\alpha = (0^{g-1}, g-1)$, we have

$$(0) \implies (0, 1) \implies (0^2, 2) \implies \dots \implies (0^{g-2}, g-2) \implies (0^{g-1}, g-1) = \alpha. \quad \text{Q.E.D.}$$

By Proposition 2.1 and the result of Eisenbud-Harris¹⁾ we get the following which is one of the main theorems in the paper 1):

Remark 2.2. Any primitive Schubert index of genus g and weight $\leq g-2$ is dimensionally proper.

Proposition 2.3. *Let α be a primitive Schubert index of genus g and weight $g-1$. If α is minimal, then g is odd and $\alpha = (0^{(g+1)/2}, 2^{(g-1)/2})$.*

Proof. Let $j_0 = \min \{j \mid \alpha_j \neq 0\}$. Since α is minimal, by Corollary 1.2 we have

$$2j_0 + 2 = a(\alpha) = \gamma(\alpha) + 1 = \alpha_{g-1} + g + 1,$$

$$\alpha_{g-2} = \alpha_{g-1} \text{ and } \alpha_i \geq 2, \text{ all } i \geq j_0.$$

Suppose that $\alpha_{g-1} > 2$. Then we have

$$g-1 = w(\alpha) = \sum_{i=j_0}^{g-3} \alpha_i + 2\alpha_{g-1} \geq 2(g-j_0-2) + 2\alpha_{g-1}$$

$$= -(2j_0+2) + 2g-2 + 2\alpha_{g-1} = g-3 + \alpha_{g-1} > g-1,$$

which is a contradiction. Hence we get $\alpha_{g-1} = 2$, which implies that $\alpha_i = 2$, all $i \geq j_0$. Since we have $g-1 = w(\alpha) = 2(g-j_0)$, the genus $g = 2j_0 - 1$ must be odd. Q.E.D.

We know that the Schubert index $(0^{(g+1)/2}, 2^{(g-1)/2})$ is dimensionally proper²⁾. Moreover, we know that the following, which is the main theorem in the paper 2), holds.

Remark 2.4. Any primitive Schubert index of genus g and weight $g-1$ is dimensionally proper.

Proposition 2.5. Let α be a primitive Schubert index of genus g and weight g . If α is minimal, then g is even and $\alpha = (0^{g/2+1}, 2^{g/2-3}, 3^2)$.

Proof. Let $j_0 = \min \{j \mid \alpha_j \neq 0\}$. Since α is minimal, by Corollary 1.2 we have

$$2j_0 + 2 = a(\alpha) = \gamma(\alpha) + 1 = \alpha_{g-1} + g + 1,$$

$$\alpha_{g-2} = \alpha_{g-1} \text{ and } \alpha_i \geq 2, \text{ all } i \geq j_0.$$

Suppose that $\alpha_{g-1} = 2$. Then we have $\alpha_i = 2$ for all $i \geq j_0$. Hence we get

$$g = w(\alpha) = 2(g-j_0) = 2g - \alpha_{g-1} - g - 1 + 2 = g - 1,$$

which is a contradiction. Hence we get $\alpha_{g-1} \geq 3$. Therefore we have

$$g = w(\alpha) = 2(g-j_0) + \sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2) + (\alpha_{g-1} - 2)$$

$$= \alpha_{g-1} + g + 1 - 2(j_0 + 1) + g - 1 + \sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2)$$

$$= g - 1 + \sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2),$$

which implies that

$$\sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2) = 1.$$

Hence we must have

$$\alpha_i - 2 = 0, \text{ all } i \text{ with } j_0 \leq i \leq g-3 \text{ and } \alpha_{g-2} - 2 = 1$$

because of $\alpha_{g-2} = \alpha_{g-1} \geq 3$. Therefore we get

$$\alpha = (0^{j_0}, 2^{g-2-j_0}, 3, 3).$$

Since we have

$$g = w(\alpha) = 2(g-2-j_0) + 3 + 3 = 2g - 2j_0 + 2,$$

the genus $g = 2j_0 - 2$ is even. Hence we get $\alpha = (0^{g/2+1}, 2^{g/2-3}, 3, 3)$. Q.E.D.

In the case of weight $g+1$ we have two types of minimal Schubert indices as follows :

Proposition 2.6. *Let α be a primitive Schubert index of genus g and weight $g+1$. If α is minimal, then one of the following holds :*

- (1) g is odd ≥ 7 and $\alpha = (0^{(g+3)/2}, 2^{(g-7)/2}, 4^2)$.
- (2) g is even ≥ 8 and $\alpha = (0^{(g+2)/2}, 2^{(g-8)/2}, 3^3)$.

Proof. Let $j_0 = \min \{j \mid \alpha_j \neq 0\}$. Since α is minimal, by Corollary 1.2 we have

$$\begin{aligned} 2j_0 + 2 &= a(\alpha) = \gamma(\alpha) + 1 = \alpha_{g-1} + g + 1, \\ \alpha_{g-2} &= \alpha_{g-1} \text{ and } \alpha_i \geq 2, \text{ all } i \geq j_0. \end{aligned}$$

If $\alpha_{g-1} = 2$, then we have $w(\alpha) = g-1$, which is a contradiction. Hence $\alpha_{g-1} \geq 3$. Therefore we have

$$\begin{aligned} g+1 &= w(\alpha) = 2(g-j_0) + \sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2) + (\alpha_{g-1} - 2) \\ &= \alpha_{g-1} + g + 1 - 2(j_0 + 1) + g - 1 + \sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2) \\ &= g - 1 + \sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2), \end{aligned}$$

which implies that

$$\sum_{i=j_0}^{g-3} (\alpha_i - 2) + (\alpha_{g-2} - 2) = 2.$$

Hence we have either

- (1) $\alpha_i = 2$ for any i with $j_0 \leq i \leq g-4$ and $\alpha_{g-3} = \alpha_{g-2} = 4$

or

- (2) $\alpha_i = 2$ for any i with $j_0 \leq i \leq g-5$ and $\alpha_{g-4} = \alpha_{g-3} = \alpha_{g-2} = 3$.

In the case (1) we must have

$$g+1 = 2(g-2-j_0) + 8 = 2g-2j_0+4,$$

which implies that $g=2j_0-3$ is odd. Hence we get $j_0 = (g+3)/2$, which implies that

$$\alpha = (0^{(g+3)/2}, 2^{(g-7)/2}, 4^2).$$

In the case (2) we must have

$$g+1 = 2(g-3-j_0) + 9 = 2g-2j_0+3,$$

which implies that $g=2j_0-2$ is even. Hence we get $j_0 = (g+2)/2$, which implies that

$$\alpha = (0^{(g+2)/2}, 2^{(g-8)/2}, 3^3). \quad Q.E.D.$$

§3. On Schubert indices α with $a(\alpha) = g(\alpha) - 1$

In the final section, using the result in Section 1 we investigate the Schubert indices α of genus g with $a(\alpha) = g-1$ satisfying the semigroup condition. Here we say that α satisfies the

semigroup condition if $N \setminus L(\alpha)$ is a subsemigroup of the additive semigroup N of non-negative integers where

$$L(\alpha) = \{\alpha_i + i + 1 \mid i = 0, 1, \dots, g-1\}.$$

Such indices satisfy the following:

Lemma 3.1. *If α is a Schubert index of genus g with $a(\alpha) = g-1$ satisfying the semigroup condition, then one of the following holds:*

- (1) α is primitive.
- (2) $\alpha = (0^{g-2}, 1, g-1)$.

Proof. Let $\alpha = (0^{g-2}, \alpha_{g-2}, \alpha_{g-1})$ be non-primitive, i.e., $2(g-1) = 2a(\alpha) \leq \alpha_{g-1} + g$. Hence we get $\alpha_{g-1} \geq g-2$. Since $\alpha_{g-1} \leq g-1$ and α satisfies the semigroup condition, we must have $\alpha_{g-1} = g-1$. Suppose that $\alpha_{g-2} > 1$. Then $g-1$ and g belong to the semigroup $H(\alpha) = N \setminus L(\alpha)$. Hence

$$L(\alpha) \ni \alpha_{g-1} + g = 2g-1 = (g-1) + g \in H(\alpha),$$

which is a contradiction. Therefore we get our desired result. Q.E.D.

Using Corollary 1.2 the minimal primitive Schubert indices α with $a(\alpha) = g(\alpha) - 1$ are determined.

Proposition 3.2. *If α is a minimal primitive Schubert index of genus g with $a(\alpha) = g-1$, then we have $g \geq 5$ and $\alpha = (0^{g-2}, g-3, g-3)$. More explicitly, if β is a primitive Schubert index of genus g with $a(\beta) = g-1$, then we have either $(0) \leq \beta$ or $(0^{h-2}, h-3, h-3) \leq \beta$ for some h with $5 \leq h \leq g$.*

Proof. Let $\alpha = (0^{g-2}, \alpha_{g-2}, \alpha_{g-1})$ be a minimal primitive Schubert index of genus g . By Corollary 1.2 we have

$$2g-2 = 2a(\alpha) = \gamma(\alpha) + 1 = \alpha_{g-1} + g + 1 \text{ and } \alpha_{g-2} = \alpha_{g-1} \geq 2.$$

Hence we get

$$\alpha_{g-2} = \alpha_{g-1} = g-3 \geq 2.$$

Let β be a primitive Schubert index of genus g with $a(\beta) = g-1$ such that $(0) \leq \beta$ does not hold. Let $\alpha = (0^{h-1}, \alpha_{h-1}) \leq \beta$. Then we have $\alpha_{h-1} \leq h-1$. But by Proposition 2.1 α is not minimal. Hence if $\alpha \leq \beta$ and if α is minimal, then we must have $\alpha = (0^{h-2}, \alpha_{h-2}, \alpha_{h-1})$. By the above we obtain $\alpha = (0^{h-2}, h-3, h-3)$ with $5 \leq h \leq g$. Q.E.D.

Now we say that a Schubert index α is *Weierstrass* if there exists a nonsingular pointed curve (C, P) such that $\alpha(P) = \alpha$, that is to say, $\mathcal{C}_\alpha \neq \emptyset$. Hence if a Schubert index α is dimensionally proper, then it is Weierstrass. Then using Lemma 3.1, Proposition 3.2 and the result of Eisenbud-Harris¹⁾ we get the following:

Corollary 3.3. *Suppose that for any $g \geq 3$ the Schubert index $(0^{g-2}, 1, g-1)$ is Weierstrass and that for any $g \geq 5$ the Schubert index $(0^{g-2}, g-3, g-3)$ is dimensionally proper, then any*

Schubert index α of genus g with $a(\alpha)=g-1$ is Weierstrass.

For a Schubert index α of genus g with $a(\alpha) \neq g-1$ we get the following results:

Remark 3.4. (1) For any $g \geq 1$ a Schubert index α of genus g with $a(\alpha)=g+1$ or g is dimensionally proper³⁾.

(2) Let $r \geq 2$. Then there exists a non-Weierstrass primitive Schubert index α of genus g with $a(\alpha)=g-r$ ⁴⁾.

References

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