Primitive numerical semigroups of cyclic index 0 starting with 11

Jiryo KOMEDA

Department of Liberal Arts

Abstract

Let H be a *numerical semigroup*, i.e., a subsemigroup of the additive semigroup $\mathbf{N}$ of non-negative integers whose complement $\mathbf{N}\backslash H$ in $\mathbf{N}$ is a finite set. In this paper, we define the cyclic index of H and show that if H is the set of non-gaps at a ramification point of a cyclic covering of $\mathbf{P}^1$ of prime degree p, then it is of cyclic index 0. Moreover, we study the converse problem in the case where H is a primitive numerical semigroup starting with 11.

Key Words: Numerical semigroup, Curve, Cyclic covering

§1. Cyclic index.

Let H be a numerical semigroup starting with $n \geq 2$. We denote by $S(H)$ the standard basis for H, i.e.,

$$S(H) = \{n, s_1, s_2, \dots, s_{n-1}\},$$

where $s_i = \mathrm{Min}\{h \in H |\ h \equiv i \bmod n\}$ for all $1 \leq i \leq n-1$. We remark that $s_i + s_{n-i} \equiv 0 \bmod n$ for all $1 \leq i \leq [n/2]$. Set

$$s = \mathrm{Min}\{s_i + s_{n-i} |\ i = 1, \dots, [n/2]\}.$$

We define the non-negative integer $Cy(H)$ to be

$$\frac{1}{n}\sum_{i=1}^{[n/2]}(s_i + s_{n-i} - s),$$

which is called the *cyclic index* of H.

Example 1.1 Let H be a numerical semigroup starting with 2 or 3. Then it follows immediately that $Cy(H) = 0$.

Example 1.2 For two integers n and m satisfying $1 \leq n \leq m \leq 3n-1$ let H be a numerical semigroup with $S(H) = \{4, s_1 = 4n+1, s_2 = 8n+2, s_3 = 4m+3\}$. Then we get $s = s_1 + s_3$, because $2s_2 - (s_1 + s_3) = 4(3n-m) > 0$. Hence, $Cy(H) = 3n - m > 0$.

Example 1.3 Let H be a numerical semigroup generated by two elements $a < b$. It is straightforward to prove that $s_i + s_{a-i} = ba$ for all $1 \leq i \leq [a/2]$. Hence, $Cy(H) = 0$.

Example 1.4 Let H be a numerical semigroup with

$$\mathbf{N}\backslash H = \{1, 2, \dots, g-1, g+i\}$$

for some $1 \leq i \leq g-1$. Then we have $Cy(H) = 1$ if $i \neq g/2$ and $Cy(H) = 2$ if $i = g/2$. In fact,

$$S(H) = \{n = g, s_1 = g+1, \ldots, s_{i-1} = g+i-1, s_{i+1} = g+i+1, \ldots, s_{g-1} = 2g-1, s_i = 2g+i\}.$$

Hence, $s = 3g$. Since $s_i + s_{g-i} = 4g$ (resp. $5g$) for $i \neq g/2$ (resp. $i = g/2$), we get the desired result.

Example 1.5 Let H be a numerical semigroup with

$$\mathbf{N} \backslash H = \{1, 2, \ldots, g-2, g, 2g-1\}.$$

Then

$$S(H) = \{n = g-1, s_2 = g+1, s_3 = g+2, \ldots, s_{g-2} = 2g-3, s_1 = 3g-2\}.$$

Consequently, $s = 3(g-1)$. Therefore, we obtain

$$Cy(H) = \frac{1}{g-1}(s_1 + s_{g-2} - s) = 2.$$

§2. The cyclic indices of ramification points of cyclic coverings of $\mathbf{P}^1$ of prime degree p.

In this paper, by a *curve* we mean a complete non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. Let C be a curve and P its point. We denote by $H(P)$ the set of integers which are pole orders at P of regular functions on $C \backslash \{P\}$. Let n be an integer≥ 2. A numerical semigroup H starting with n is said to be *n-cyclic* if there exists a cyclic covering $\pi : C \longrightarrow \mathbf{P}^1$ of curves with a total ramification point P such that $H(P) = H$, where $\mathbf{P}^1$ denotes the projective 1-space over k. In this section, we shall show the following theorem :

Theorem 2.1 *Let p be a prime number and H a p-cyclic numerical semigroup. Then $Cy(H) = 0$.*

Proof. Let $\pi : C \longrightarrow \mathbf{P}^1$ be a cyclic covering of curves of degree p with a ramification point P such that $H(P) = H$. It follows from the proof of Proposition 1 in the paper 1) that

$$S(H(P)) = \left\{p, b, -p\sum_{q=1}^{p-1}\left[\frac{-mq}{p}\right] i_q - mb \ (m = 1, 2, \ldots, p-2)\right\},$$

where $i_1, i_2, \ldots, i_{p-1}$ are some non-negative integers with $p \not| \sum_{q=1}^{p-1} q i_q$ and $b = \sum_{q=1}^{p-1} q i_q$. Hence, we get

$$\left\{s_i + s_{p-i} \mid i = 1, 2, \ldots, \left[\frac{p}{2}\right]\right\} = \left\{-p\sum_{q=1}^{p-1}\left[\frac{-q}{p}\right] i_q - b + b\right\}$$

$$\bigcup \left\{-p\sum_{q=1}^{p-1}\left[\frac{-mq}{p}\right] i_q - mb + \left(-p\sum_{q=1}^{p-1}\left[\frac{-(p-m)q}{p}\right] i_q - (p-m)b\right) \mid m = 2, 3, \ldots, \left[\frac{p}{2}\right]\right\}$$

We find that for all $2 \leq m \leq [p/2]$,

$$-p\sum_{q=1}^{p-1}\left[\frac{-mq}{p}\right] i_q - mb + \left(-p\sum_{q=1}^{p-1}\left[\frac{-(p-m)q}{p}\right] i_q - (p-m)b\right)$$

$$= -p\sum_{q=1}^{p-1}\left(\left[\frac{-mq}{p}\right] + \left[\frac{-(p-m)q}{p}\right]\right) i_q - pb$$

$$= -p\sum_{q=1}^{p-1}(-1-q)i_q - pb = p\sum_{q=1}^{p-1} i_q = -p\sum_{q=1}^{p-1}\left[\frac{-q}{p}\right] i_q,$$

which shows that $Cy(H) = 0$. *Q.E.D.*

In order to study the converse of the above theorem in the next section we need the following remark.

Remark 2.2 Let p be a prime number and H a numerical semigroup starting with p. Then the following conditions are equivalent.

i) H is p-cyclic.

ii) $S(H) = \left\{p, \sum_{q=1}^{p-1}\left(lq - \left[\frac{lq}{p}\right]p\right) i_q \ (l = 1, 2, \ldots, p-1)\right\}$ for some non-negative integers $i_1, i_2, \ldots, i_{p-1}$.

Proof. For all $1 \leq m \leq p-1$ we have

$$(p-m)q - \left[\frac{(p-m)q}{p}\right]p + p\left[-\frac{mq}{p}\right] + mq$$

$$= pq - \left(q + \left[-\frac{mq}{p}\right]\right)p + p\left[-\frac{mq}{p}\right] + mq = 0.$$

Hence, by the proof of Proposition 1 in the paper 1) we get the desired result. *Q.E.D.*

§3. The converse problem in the case $p = 11$.

We shall study the converse of Theorem 2.1 with $p = 11$. In the case $p < 11$ this statement holds[2),3)]. In this section, we shall determine all primitive numerical semigroups of cyclic index 0 starting with 11, where a numerical semigroup H starting with n is said to be *primitive* if the largest integer in $\mathbf{N}\backslash H$ is less than $2n$. Moreover, we consider the question whether these semigroups are 11-cyclic.

Lemma 3.1 *Let H be a primitive numerical semigroup of cyclic index* 0 *starting with* 11. *Then one of the following holds*

i) $S(H) = \{11, 12, 13, \ldots, 21\}$.

ii) $S(H) = \{11, 23, 24, \ldots, 32\}$.

iii) $s = 44$, *where s is as in* §1.

Proof. We follow the notation of §1. Suppose that $s \geq 66$. Then there exists some $1 \leq i \leq 10$ such that $s_i > 33$. Hence, $H \not\ni s_i - 11 > 22 = 2 \times 11$, which shows that H is non-primitive. Consequently, we get $s = 33$ or 44 or 55. Let $s = 33$. It is straightfoward to show that i) holds. Let $s = 55$. Suppose that there exists some i such that $s_i \neq 22 + i$. Then we get $s_i = 11 + i$ or $33 + i$. If $s_i = 11 + i$, then $s_{11-i} = 44 - i$, which implies that the largest integer in $\mathbf{N}\backslash H$ is larger than or equal to $33 - i > 22$. This is a contradiction. If $s_i = 33 + i$, in a similar way to the above this is impossible. Accordingly, H satisfies ii). *Q.E.D.*

In the remainder of this section we use not only the notation in §1, but also the following. Let H be a primitive numerical semigroup of cyclic index 0 starting with 11. We set

$$t_1 = \mathrm{Min}\{s_i |\ i = 1, 2, \ldots, 10\}.$$

For all $1 \leq i \leq 10$, let $t_i \in S(H)$ such that $t_i \equiv it_1 \bmod 11$. By Remark 2.2, H is 11-cyclic if and only if the following system (A) of linear equations has a solution $(i_1, i_2, \ldots, i_{10})$ such that $i_1, i_2, \ldots, i_{10}$ are non-negative integers:

$$
\text{(A)} \begin{cases} 11(i_1+i_2+i_3+i_4+i_5+i_6+i_7+i_8+i_9+i_{10}) = s \\ i_1+2i_2+3i_3+4i_4+5i_5+6i_6+7i_7+8i_8+9i_9+10i_{10} = t_1 \\ 2i_1+4i_2+6i_3+8i_4+10i_5+i_6+3i_7+5i_8+7i_9+9i_{10} = t_2 \\ 3i_1+6i_2+9i_3+i_4+4i_5+7i_6+10i_7+2i_8+5i_9+8i_{10} = t_3 \\ 4i_1+8i_2+i_3+5i_4+9i_5+2i_6+6i_7+10i_8+3i_9+7i_{10} = t_4 \\ 5i_1+10i_2+4i_3+9i_4+3i_5+8i_6+2i_7+7i_8+i_9+6i_{10} = t_5 \end{cases}
$$

We shall investigate whether H is 11-cyclic, that is, whether the system (A) has a solution consisting of non-negative integers.

i) $S(H)=\{11,12,13,\ldots,21\}$. Then (A) has a solution $(1,0,0,0,1,1,0,0,0,0)$.

ii) $S(H)=\{11,23,24,\ldots,32\}$. Then (A) also has a solution $(1,0,0,0,2,2,0,0,0,0)$.

iii) Let $s=44$. We denote by $I(H)$ the subset of $\{1,2,3,4,5\}$ such that if $i\in I(H)$, then $s_i=11+i$ and $s_{11-i}=33-i$, and otherwise, $s_i=22+i$ and $s_{11-i}=22-i$. The cardinality of the set of subsets of the set $\{1,2,3,4,5\}$ is 32. By computation we obtain the following table for solutions of the system (A) in each case.

	$I(H)$	$S(H)\backslash\{11\}$	a solution consisting of non-negative integers
(1)	$\emptyset$	$\{17,18,19,20,21,23,24,25,26,27\}$	$(1,0,0,0,2,1,0,0,0,0)$
(2)	$\{1\}$	$\{12,17,18,19,20,24,25,26,27,32\}$	$(0,2,1,0,1,0,0,0,0,0)$
(3)	$\{2\}$	$\{13,17,18,19,21,23,25,26,27,31\}$	$(1,0,1,1,1,0,0,0,0,0)$
(4)	$\{3\}$	$\{14,17,18,20,21,23,24,26,27,30\}$	$(1,1,0,0,1,1,0,0,0,0)$
(5)	$\{4\}$	$\{15,17,19,20,21,23,24,25,27,29\}$	$(1,1,0,0,1,0,1,0,0,0)$
(6)	$\{5\}$	$\{16,18,19,20,21,23,24,25,26,28\}$	$(0,1,1,0,1,1,0,0,0,0)$
(7)	$\{1,2\}$	$\{12,13,17,18,19,25,26,27,31,32\}$	$(1,2,0,0,0,0,1,0,0,0)$
(8)	$\{1,3\}$	$\{12,14,17,18,20,24,26,27,30,32\}$	$(1,1,0,1,1,0,0,0,0,0)$
(9)	$\{1,4\}$	$\{12,15,17,19,20,24,25,27,29,32\}$	$(0,1,2,1,0,0,0,0,0,0)$
(10)	$\{1,5\}$	$\{12,16,18,19,20,24,25,26,28,32\}$	No
(11)	$\{2,3\}$	$\{13,14,17,18,21,23,26,27,30,31\}$	$(1,1,0,0,2,0,0,0,0,0)$
(12)	$\{2,4\}$	$\{13,15,17,19,21,23,25,27,29,31\}$	$(2,0,0,0,1,1,0,0,0,0)$
(13)	$\{2,5\}$	$\{13,16,18,19,20,21,23,25,26,28\}$	$(0,1,2,0,1,0,0,0,0,0)$
(14)	$\{3,4\}$	$\{14,15,17,20,21,23,24,27,29,30\}$	$(2,0,0,0,1,0,1,0,0,0)$
(15)	$\{3,5\}$	$\{14,16,18,20,21,23,24,26,28,30\}$	$(0,1,1,1,1,0,0,0,0,0)$
(16)	$\{4,5\}$	$\{15,16,19,20,21,23,24,25,28,29\}$	$(1,0,1,0,1,1,0,0,0,0)$
(17)	$\{1,2,3\}$	$\{12,13,14,17,18,26,27,30,31,32\}$	$(2,1,0,0,0,0,0,1,0,0)$
(18)	$\{1,2,4\}$	$\{12,13,15,17,19,25,27,29,31,32\}$	$(1,1,1,0,0,1,0,0,0,0)$
(19)	$\{1,2,5\}$	$\{12,13,16,18,19,25,26,28,31,32\}$	No
(20)	$\{1,3,4\}$	$\{12,14,15,17,20,24,27,29,30,32\}$	$(1,0,1,2,0,0,0,0,0,0)$
(21)	$\{1,3,5\}$	$\{12,14,16,18,20,24,26,28,30,32\}$	$(2,0,0,0,2,0,0,0,0,0)$
(22)	$\{1,4,5\}$	$\{12,15,16,19,20,24,25,28,29,32\}$	$(1,0,2,0,1,0,0,0,0,0)$
(23)	$\{2,3,4\}$	$\{13,14,15,17,21,23,27,29,30,31\}$	No
(24)	$\{2,3,5\}$	$\{13,14,16,18,21,23,26,28,30,31\}$	No
(25)	$\{2,4,5\}$	$\{13,15,16,19,21,23,25,28,29,31\}$	$(1,1,1,0,0,0,1,0,0,0)$
(26)	$\{3,4,5\}$	$\{14,15,16,20,21,23,24,28,29,30\}$	$(1,0,1,0,2,0,0,0,0,0)$
(27)	$\{1,2,3,4\}$	$\{12,13,14,15,17,27,29,30,31,32\}$	$(2,0,0,1,0,1,0,0,0,0)$
(28)	$\{1,2,3,5\}$	$\{12,13,14,16,18,26,28,30,31,32\}$	No
(29)	$\{1,2,4,5\}$	$\{12,13,15,16,19,25,28,29,31,32\}$	$(2,0,1,0,0,0,1,0,0,0)$
(30)	$\{1,3,4,5\}$	$\{12,14,15,16,20,24,28,29,30,32\}$	No
(31)	$\{2,3,4,5\}$	$\{13,14,15,16,21,23,28,29,30,31\}$	No
(32)	$\{1,2,3,4,5\}$	$\{12,13,14,15,16,28,29,30,31,32\}$	$(3,0,0,0,0,0,0,0,1,0)$

For example, let H be as in the case (10). Then we consider the system (A) of linear equations

$$\begin{cases} 11(i_1+i_2+i_3+i_4+i_5+i_6+i_7+i_8+i_9+i_{10}) = 44 \\ i_1+2i_2+3i_3+4i_4+5i_5+6i_6+7i_7+8i_8+9i_9+10i_{10} = 12 \\ 2i_1+4i_2+6i_3+8i_4+10i_5+i_6+3i_7+5i_8+7i_9+9i_{10} = 24 \\ 3i_1+6i_2+9i_3+i_4+4i_5+7i_6+10i_7+2i_8+5i_9+8i_{10} = 25 \\ 4i_1+8i_2+i_3+5i_4+9i_5+2i_6+6i_7+10i_8+3i_9+7i_{10} = 26 \\ 5i_1+10i_2+4i_3+9i_4+3i_5+8i_6+2i_7+7i_8+i_9+6i_{10} = 16 \end{cases}$$

If we set $i_7 = i_8 = i_9 = i_{10} = 0$, then the above system becomes

$$\begin{cases} 11(i_1+i_2+i_3+i_4+i_5+i_6) = 44 \\ i_1+2i_2+3i_3+4i_4+5i_5+6i_6 = 12 \\ 2i_1+4i_2+6i_3+8i_4+10i_5+i_6 = 24 \\ 3i_1+6i_2+9i_3+i_4+4i_5+7i_6 = 25 \\ 4i_1+8i_2+i_3+5i_4+9i_5+2i_6 = 26 \\ 5i_1+10i_2+4i_3+9i_4+3i_5+8i_6 = 16 \end{cases}$$

Then by calculation this system has a unique solution $(1,1,1,-1,2,0)$. Moreover, it is easy to see that the system (A) has the solutions

$$i_1 = 1+i_{10},\ i_2 = 1+i_9,\ i_3 = 1+i_8,\ i_4 = -1+i_7,\ i_5 = 2-i_7-i_8-i_9-i_{10} \text{ and } i_6 = -i_7-i_8-i_9-i_{10}$$

where i_7, i_8, i_9 and i_{10} are arbitrary. Therefore, the system (A) has no solutions consisting of non-negative integers.

References

1) J. Komeda : Numerical semigroups and non-gaps of Weierstrass points, Research Reports of Ikutoku Technical University B-9 (1985), 89-94.

2) I. Morrison and H. Pinkham : Galois Weierstrass points and Hurwitz characters, Ann. Math., 124 (1984) 591-625

3) J. Komeda : Numerical semigroups of cyclic index 0, In preparation.