

# Daubechies Wavelet and Gibbs Phenomenon

Yasuo Tachibana

Dept. of Electric and Electronic

## Abstract

In this paper it is shown that Gibbs phenomenon happens in the finite expansion series of the Daubechies Wavelet. And its application to the compensation filter for missed observation is presented. On the base of this property it is discussed that the compensation method for the signal with the missed observation without any prediction procedure. The proposed method is able to detect the transition points between the normal and missed states. And a filter is proposed which can detect whether signal exists in the normal state or missed state, compensates the missed part of it, reduces the noise in it and outputs the filtered signal with constant delay. By simulation the appropriateness and effectiveness of the proposed filter are shown.

**Key Words** Fourier Analysis, Wavelet, Gibbs Phenomenon, Digital Processing, Missed Observation, Compensation Filter

## 1. Introduction

In this paper it is shown that Gibbs phenomenon happens in the finite expansion series of the Daubechies Wavelet. And for its application a construction method of a compensation filter for the missed data is proposed. It is based on the Gibbs phenomenon, which happens on the finite expansion series of the Daubechies Wavelet. There are many cases in which we have to carry out the compensation of missed observation signals. For examples, signal defects randomly generated on several measurements, lost data by blinding on the Doppler radar measurement and communication reports at random arrivals. In this paper we treat the filtering of signal with missed observation [3]-[9]. For the studies of this area, R.H. Jones [4] and R.E.Parzen [5] discussed it for the frequency deviation of the Doppler effect of radar measurement. P.A.Sheinrock [6] treated it also for the case of Bernoulli process. T.Sakai treated it by the AR model fitting [7]. The  $\varepsilon$ -separation filter [10] which is nonlinear filter can be used for the area discussed in this paper. One of the authors has been shown that a

compensation filter for the missed data can be constructed by the ratio of two adequate FIR filters [3]. And its several characteristics have been presented. Further more the author has been proposed another compensation filter for missed data which generated by the finite Fourier series fitting for the normal part of signal and was used for the communication signal restoration. If we can find the structure of the model of signal we can construct the compensation filter for the missed data by Kalman filter [9]. In this paper we intend to construct the filter which is satisfied the following three conditions.

[A] Detection of the missed states without any prediction

[B] Ability for the compensation of the missed part of the signal, and filtering with constant delay

[C] Noise reduction for the whole time

The filter proposed by one of the authors does not satisfy the condition [A] because it is assumed that the points of state changes are given. As the method discussed in [4]~[7] are carried out the spectrum analysis without any compensation, they never contain the

compensation process. As the compensation method using Kalman filter contains the prediction based on the signal model, the condition [A] is not satisfied. The main purpose of the  $\varepsilon$  - separation filter does not focus to the compensation of the signals. If we can find the point where the missed phenomenon occurs, the conventional filters by the curve fitting for the observation signal such as trigonometric or finite Fourier transform interpolations are used for the compensation. By the way it has been shown that the finite expansion series of the Daubechies Wavelet has the Gibbs phenomenon [1][2]. In this paper we would like to show that we are able to construct a compensation filter which detects the points for the happenings of missed observation without any prediction by using this phenomenon. Namely this procedure satisfies the condition [A]. By the adaptive linear curve fitting with this detection procedure we can construct a filter satisfies the conditions [A], [B] and [C].

At the first Wavelet was discovered by Y.Meyer in the analysis of geological signal. S.G. Mallat established the concept of the multiresolution analysis (approximation) and showed clearly the fundamental characteristics of the discrete Wavelet expansion. The multiresolution analysis provides a sequence of the monotone linear sub spaces in a Hilbert space. There are two special functions in the multiresolution analysis, that is, scaling function and wavelet function. Inversely it is said that a scaling function (or wavelet function) gives a multiresolution analysis. We can get the orthogonal bases of the monotone subspaces by the scaling function and also we get the orthonormal bases of the subspaces, which are the differences of the adjacent linear subspaces. Usually the support of these functions is not

compact. At 1988 I.Daubechies discovered a Wavelet with compact support Wavelet. We can treat the finite Wavelet expansion clearly by Daubechies Wavelet. Recently it was pointed out that the finite expansion of Daubechies Wavelet appeared the Gibbs phenomenon, which is usually observed in the finite Fourier expansion. The difference between the observed signal and the finite Daubechies Wavelet expansion shows the sharp peaks at the discontinuous points. By the property discussed above, we can detect the beginning and ending points of the missed Observation State. Adding this detection procedure we use a linear curve fitting for the normal observation parts of signal in order to get the filter satisfying the conditions [A]  $\sim$  [C]. For several types of signal the responses of the proposed filter are presented in order to show the appropriateness and the effectiveness.

## 2. The finite Wavelet expansion

We would like to give the fundamental concepts of the finite Wavelet expansion which are need for the descriptions discussed later. Let  $\mathbf{R}$  be the whole real number,  $\mathbf{Z}$  be the whole integer,  $L^2(\mathbf{R})$  be a Hilbert space in  $\mathbf{R}$ , and  $\{V_j\}_{j \in \mathbf{Z}}$  be a sequence of the monotone linear subspaces in  $\mathbf{R}$ . If  $\{V_j\}_{j \in \mathbf{Z}}$  satisfies the following five conditions, it is called as the multi resolution analysis by S.G.Mallat<sup>[12]</sup>. That is,

- 1°  $V_p \subset V_{p+1} \quad p \in \mathbf{Z}$
- 2°  $\bigcup_{p=-\infty}^{\infty} V_p$  is dense in  $L^2(\mathbf{R})$   $\bigcap_{p=-\infty}^{\infty} V_p = \{0\}$
- 3°  $f(x) \in V_p \Leftrightarrow f(2x) \in V_{p+1} \quad \forall p \in \mathbf{Z}$
- 4°  $f(x) \in V_p \Rightarrow f(x - 2^{-p}q) \in V_p \quad \forall p, q \in \mathbf{Z}$
- 5° There exists an isomorphism  $I$  from  $V_0$  onto  $L^2(\mathbf{Z})$  which commutes with the action of  $\mathbf{Z}$ .

where  $L^2(\mathbf{Z})$  is the Hilbert space constructed by square-summable sequences  $\{V_p\}_{p \in \mathbf{Z}}$ . There exists a unique function  $\phi(x)$  for a multiresolution analysis, which is called as scaling function. For any  $p \in \mathbf{Z}$ ,  $\{\sqrt{2^p} \phi(2^p x - q)\}_{q \in \mathbf{Z}}$  is an Orthonormal basis of  $V_p$ . Where  $p$  is called as dilation (scale), also

$q$  is called as translation (shift). Property 2° of a multiresolution analysis shows

$$\sqrt{2} \phi(x/2) \in V_{-1} \subset V_0 \quad (1)$$

$$\sqrt{2} \phi(x/2) = \sum_{q=-\infty}^{\infty} h_q \phi(x - q). \quad (2)$$

That is, if  $\{h_q\}_{q \in \mathbf{Z}}$  satisfies the condition

$$|h_q| = O(1 + q^2)^{-1} \quad (3)$$

then  $\phi(x)$  is derived by this sequence [12]. Let  $\hat{\phi}(\omega)$  be the Fourier transform of  $\phi(x)$ , and let

$$H(\omega) = \frac{1}{\sqrt{2}} \sum_{q=-\infty}^{\infty} h_q e^{-i\omega q} \quad (4)$$

be the Fourier transform of  $\{h_q\}_{q \in \mathbf{Z}}$ . By the property 2° we get

$$\hat{\phi}(2\omega) = H(\omega) \hat{\phi}(\omega). \quad (5)$$

Next we give the following functions

$$G(\omega) = e^{-i\omega} \overline{H(\omega + \pi)} \quad (6)$$

$$\hat{\psi}(\omega) = G(\omega/2) \hat{\phi}(\omega/2), \quad (7)$$

where  $i = \sqrt{-1}$ . The function  $\psi(x)$  the inverse Fourier transform of  $\hat{\psi}(\omega)$  is called as the wavelet function. Usually  $\phi(x)$  and  $\psi(x)$  are together called as Wavelet. Let  $W_p$  be the Orthonormal complement  $V_p$  in  $V_{p+1}$ , that is

$$V_{p+1} = V_p \oplus W_p. \quad (8)$$

Then for different  $p$  and  $p'$  the linear spaces  $W_p$  and  $W_{p'}$  are orthogonal. And

$\{\sqrt{2^p} \psi(2^p x - q)\}_{q \in \mathbf{Z}}$  is an Orthonormal base of

$W_p$ . For any integer  $M_0$ , the following relation is satisfied

$$L^2(\mathbf{R}) = V_{M_0} \oplus \left( \bigoplus_{m=M_0}^{\infty} W_m \right). \quad (9)$$

Therefore arbitrary  $\forall f(x) \in L^2(\mathbf{R})$  is expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} \left\langle f(u), \sqrt{2^{M_0}} \phi(2^{M_0} u - n) \right\rangle \sqrt{2^{M_0}} \phi(2^{M_0} x - n) \\ + \sum_{m=M_0}^{\infty} \sum_{n=-\infty}^{\infty} \left\langle f(u), \sqrt{2^m} \psi(2^m u - n) \right\rangle \sqrt{2^m} \psi(2^m x - n). \quad (10)$$

Where  $\langle * \rangle$  indicates the inner product. For Daubechies Wavelet  $\phi(x)$  and  $\psi(t)$  are defined by a parameter  $N (= 2, 3, \dots)$ . The support of  $\phi(x)$  is  $[0, 2N - 1]$ , and that of  $\psi(t)$  is  $[-(N - 1), N]$ . Next let  $\Lambda_N$  be the set of functions in  $L^2(\mathbf{R})$  which support are contained in  $[0, 2N - 1]$ . We define

$$n\ell 0 = -2(N - 1) \quad nh0 = 2^{M_0}(2N - 1) - 1 \quad (11)$$

$$n\ell(m) = -(N - 1) \quad nh(m) = 2^m(2N - 1) + N - 2. \quad (12)$$

Then for  $f(x) \in \Lambda_N$  we set such that the  $n$  appeared in the first term of (10) is limited to  $n\ell 0 \sim nh0$ , also  $n$  appeared in the second term of (10) is limited to  $n\ell(m) \sim nh(m)$ , and also  $m$  is limited to  $M_0 \sim M$ .

$$f_M(x) = \sum_{n=n\ell 0}^{nh0} \left\langle f(u), \sqrt{2^{M_0}} \phi(2^{M_0} u - n) \right\rangle \sqrt{2^{M_0}} \phi(2^{M_0} x - n) \\ + \sum_{m=M_0}^M \sum_{n=n\ell(m)}^{nh(m)} \left\langle f(u), \sqrt{2^m} \psi(2^m u - n) \right\rangle \sqrt{2^m} \psi(2^m x - n) \quad (13)$$

Hereafter we would like to use the finite expansion  $f_M(x)$  of a function  $f(x)$ . In order to calculate the value of  $\phi(x)$  the subdivision method by C.A. Michelli is introduced in [13]. When we use the Daubechies Wavelet the right hand side of the formula (2) becomes a finite series. The values  $\phi(0), \phi(1), \dots, \phi(2N - 2)$  are defined by eigen value vectors of a matrix constructed the parameters  $h_0, h_1, \dots, h_{2N-1}$ . Next by (2) we can derive

$$\phi\left(\frac{j}{2^L}\right) \quad (j = 0, 1, \dots, N_H) \quad (14)$$

iteratively. Where  $N_H$  is given by

$$N_H = 2^L (2N - 1). \quad (15)$$

Where  $L$  is a non-negative integer. The following relation is derived from (6).

$$\psi(x) = \sqrt{2} \sum_{k=0}^{2N-1} (-1)^k h_k \phi(2x + k - 1). \quad (16)$$

And from the above relation we can get  $\psi(\frac{j}{2^L} - (N - 1))$  ( $j = 0, 1, \dots, N_H$ ). Therefore we can

get  $f_M(\frac{j}{2^L})$  ( $j = 0, 1, \dots, N_H$ ) with the resolution

$\frac{1}{2^L}$  by the following equations.

$$\langle f(u), \sqrt{2^p} \phi(2^p u - q) \rangle = \frac{\sqrt{2^p}}{2^L} \sum_{\ell=0}^{N_H} f(\frac{\ell}{2^L}) \phi(2^p \frac{\ell}{2^L} - q) \quad (17)$$

$$\langle f(x), \sqrt{2^p} \psi(2^p x - q) \rangle = \frac{\sqrt{2^p}}{2^L} \sum_{\ell=0}^{N_H} f(\frac{\ell}{2^L}) \psi(2^p \frac{\ell}{2^L} - q) \quad (18)$$

Figure 1 shows  $\phi(x)$  and  $\psi(x)$  for some parameters.

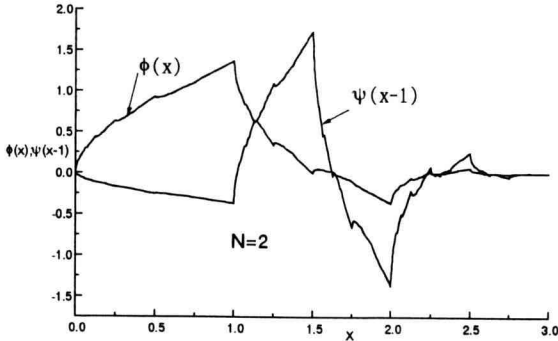


Fig.1 Scaling and Wavelet functions

### 3. Calculation of Daubechies Wavelet

The sequence  $\{\phi(x - k)\}_{k \in \mathbb{Z}}$  of the functions is a complete normal orthogonal base of  $V_0$ . And the following relation is satisfied

$$\int_0^{2N-1} \phi(x) dx = 1. \quad (19)$$

$V_0 \subset V_1$  is satisfied, therefore  $\phi(x) \in V_0$  is also contained in  $V_1$ . This means

$$\phi(x) = \sqrt{2} \sum_{k=0}^{2N-1} h_k \phi(2x - k) \quad (20)$$

The integration of  $\phi(x)$  shows the relation

$$\sum_{k=0}^{2N-1} h_k = 1/\sqrt{2}. \text{ Daubechies gave the sequences}$$

$\{h_k\}_{k=0}^{2N-1}$  for  $N = 2, \dots, 10$  with 12 digits accuracy. Generally  $\{h_k\}_{k=0}^{2N-1}$  is given by the Fourier transform of them as follows.

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{k=0}^{2N-1} h_k e^{ik\xi} = \left[ \frac{1}{2} (1 + e^{i\xi}) \right]^N Q(e^{i\xi}) \quad (21)$$

$$|Q(e^{i\xi})|^2 = \sum_{k=0}^{2N-1} \frac{(N-1+k)!}{(N-1)!k!} \sin^{2k} \left( \frac{\xi}{2} \right) \quad (22)$$

The right hand side of (22) is positive real function. Therefore there exists some polynomial  $Q(e^{i\xi})$  of  $e^{i\xi}$  by the theorem about the positive real function. The root of  $Q(e^{i\xi})$  is minimal phase, which means that the zero points of  $Q(e^{i\xi})$  are contained in the unit circle. Still more  $Q(e^{i\xi})$  is expressed only by the positive frequency. By these conditions,  $Q(e^{i\xi})$  is determined uniquely. For  $N = 4$  and  $N = 12$ ,  $\{h_k\}_{k=0}^{2N-1}$  are calculated by this procedure  $\{h_k\}_{k=0}^{2N-1}$  are given in the Table 1 with 19 digits accuracy. Using  $\{h_k\}_{k=0}^{2N-1}$  we can get  $\phi(\ell/2^L)$  ( $\ell = 0, 1, \dots, N_T$ ) from (20). Where,  $N_T = 2^L (2N - 1)$  and  $L$  is some non-negative integer which defines the resolution of the calculation. At first, we calculate  $\phi(k)$  ( $k = 1, 2, \dots, 2N - 2$ ). They are not all zero at the same time. As  $\phi(x)$  is continuous functions and satisfies (20), if they are zero at the integer points then  $\phi(x) = 0$  is derived. Still more we get  $(1 - \sqrt{2}h_0)\phi(0) = 0$ ,  $(1 - \sqrt{2}h_{2N-1})\phi(2N - 1) = 0$  from (20). At least we have known that  $1 \neq \sqrt{2}h_0$ ,  $1 \neq \sqrt{2}h_{2N-1}$  for  $N = 2, \dots, 12$ , then  $\phi(0) = 0$ ,  $\phi(2N - 1) = 0$  are derived. Also, we get a set of the simultaneous linear homogeneous equations. This set of equations has the solution, which are not zeroing at the same time. Thus  $(2N - 3)$ 's equations from the set of the

simultaneous equations are linearly independent. By solving the equations  $\phi(k)$  ( $k = 1, 2, \dots, 2N-3$ ) are expressed by  $\phi(2N-2)$ . Also  $\phi(\ell/2^L)$  ( $\ell = 0, 1, \dots, N_T$ ) are expressed by  $\phi(2N-2)$  by (20). The one ambiguity left is determined by the equation (19), such as.

$$1 = \int_0^{2^{N-1}} \phi(x) dx \approx \frac{1}{2^L} \sum_{\ell=0}^{N_T} \phi\left(\frac{\ell}{2^L}\right) \quad (23)$$

Thus we get  $\phi(\ell/2^L)$  ( $\ell = 0, 1, \dots, N_T$ ) completely. In [14], there is a sequence of functions, which converges to  $\phi(x)$ . In this paper we use the following identify which is derived by the fact of  $\phi(x)$  the support of is  $[0, 2N-1]$ .

$$\frac{1}{2}\phi\left(\frac{j}{2}\right) = \sum_{k=0}^{2N-1} h_k \phi(x-k). \quad (24)$$

From (24) we get immediately

$$(1-2h_0)\phi(0) = 0 \quad (25)$$

$$(1-2h_{2N-1})\phi(2N-1) = 0. \quad (26)$$

At least for  $N = 2 \sim 12$  we can show

$$1-2h_0 \neq 0 \quad 1-2h_{2N-1} \neq 0 \quad (27)$$

therefore we get clearly

$$\phi(0) = 0 \quad \phi(2N-1) = 0. \quad (28)$$

Also for  $k \in \mathbb{Z}$  we get

$$\phi(k) = 0 \quad (k \leq 0, k \geq 2N-1). \quad (29)$$

By (28) and (29) the equations (24) are the homogeneous equations in which the number of the unknown parameters is  $2N-2$  and the number of the equations is  $2N-2$ . Here  $\phi(x)$  is not 0 identically and is continuous function, therefore (24) shows that all of the values of  $\phi(x)$  at integers are not zero. Thus  $2N-2$ 's equations expressed by (24) are linearly dependent, so the solution of them are derived by the  $2N-3$ 's independent equations. Let us define

$$\mathbf{A} = 2 \begin{bmatrix} h_1 & h_0 & 0 & \cdots & \vdots \\ h_3 & h_2 & h_1 & h_0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{2N-1} & h_{2N-2} & \cdots & h_3 & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots \\ \vdots & h_{2N-1} & h_{2N-3} & h_{2N-3} & \vdots \end{bmatrix} \leftarrow N^{(30)}$$

$$\mathbf{F} = [\phi(1) \quad \phi(2) \quad \cdots \quad \phi(2N-3)]^T \quad (31)$$

$$\mathbf{b} = 2[0 \quad \cdots \quad 0 \quad h_0 \quad h_2 \quad \cdots \quad h_{2N-4}]^T \quad (32)$$

for the convention. The first  $2N-3$ 's equations in (24) are shown by

$$(\mathbf{I} - \mathbf{A})\mathbf{F} = \mathbf{b}\phi(2N-2) \quad (33)$$

where  $T$  shows the transpose of a matrix. At least  $N = 2 \sim 12$ ,  $\mathbf{I} - \mathbf{A}$  is non singular, thus we get

$$\mathbf{F} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}\phi(2N-2). \quad (34)$$

In (33) if we put  $\phi(2N-2) = 1$ , then we get a solution  $\{\phi'(j)\}_{j=1}^{2N-2}$ . It is different the constant multiple from the true solution  $\{\phi(j)\}_{j=1}^{2N-2}$ . The multiple constant is derived from one more equation in (24).

$$\phi'(2N-2) = 2\{h_{2N-2}\phi'(2N-2) + h_{2N-1}\phi'(2N-3)\} \quad (35)$$

By (34) and (35) we get

$$\phi\left(\frac{j}{2^L}\right) = \frac{2^L \phi'\left(\frac{j}{2^L}\right)}{\sum_{i=0}^{2^{L(2N-1)-1}} \phi'\left(\frac{i}{2^L}\right)} \quad (36)$$

#### 4. Gibbs phenomenon

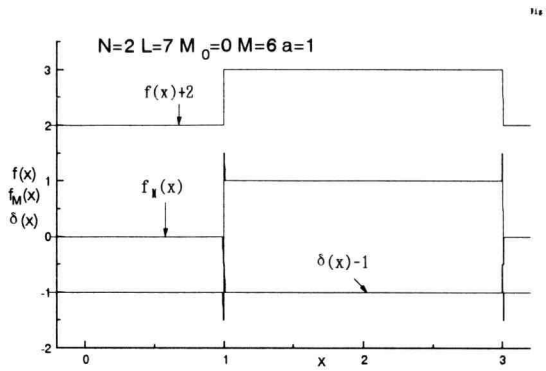
We would like to show the result derived from the examples in numerical calculations. Let  $f(x)$  be a discontinuous functions such as

**Table 1** Parameters  $h_k$

$k$	$h_k \quad N = 4$
0	0.23037781330889650080
1	0.71484657055291564680
2	0.63088076792985890790
3	-0.02798376941685985394
4	-0.18703481171909308400
5	0.03084138183556076361
6	0.03288301166688519972
7	-0.01059740178506903210
$k$	$h_k \quad N = 12$
0	0.01311225795722903193
1	0.10956627282118145830
2	0.37735513521420186840
3	0.65719872257929450860
4	0.51588647842781905780
5	-0.04476388565375193512
6	-0.31617845375277203590
7	-0.02377925725608232192
8	0.18247860592756713170
9	0.00535956967436017652
10	-0.09643212009649691717
11	0.01084913025581962717

12	0.04154627749508002170
13	-0.01221864906974727820
14	-0.01284082519829930504
15	0.00671149900879514288
16	0.00224860724099498515
17	-0.00217950361862764737
18	0.00000654512821252440
19	0.00038865306282091027
20	-0.00008850410920820031
21	-0.00002424154575702923
22	0.00001277695221937907
23	-0.00000152907175806846

$$f(x) = \begin{cases} 0 & x < a \quad x > 2N-1 \\ 1 & a \leq x \leq 2N-1 \quad (0 < a < 2N-1). \end{cases} \quad (37)$$



**Fig.2 Gibbs' phenomenon**

The shape of the finite Wavelet expansion of  $f(x)$  becomes in Figure 2 in which parameter are set as  $N = 2, L = 7, M_0 = 0, M = 6, a = 1$ .  $f_M(x)$  shows the peaks as whiskers at  $x = a$ . This phenomenon is resembled to the Gibbs phenomenon appeared in the Fourier series expansions. This has been reported in [1], [2]. All of Wavelets never show this phenomenon. There is some example of Wavelet which does not show such phenomenon. We can modify  $f_M(x)$  as the following formula.

$$f_M(x) = \int_0^{2N-1} f(u) \Phi_{M_0, M}(x, u) du \quad (38)$$

$$\begin{aligned} \Phi_{M_0, M}(x, y) &= \sum_{n=nl(0)}^{nh(0)} 2^{M_0} \phi(2^{M_0} x - n) \phi(2^{M_0} y - n) \\ &+ \sum_{m=M_0}^M \sum_{n=nl(m)}^{nh(m)} 2^m \psi(2^m x - n) \psi(2^m y - n) \\ &= \sum_{n=nl(0)}^{2^{M+1}(2N-1)-1} 2^{M+1} \phi(2^{M+1} x - n) \phi(2^{M+1} y - n) \end{aligned} \quad (39)$$

When  $M \rightarrow \infty$  it is anticipated that the support of it tends to become  $x = y$ . It was pointed out in [2]  $\lim_{M \rightarrow \infty} \Phi_{M_0, M}(x, y) = \delta(x - y)$

formally using Dirac Delta function  $\delta(x)$ . But correct proof is necessary to this conclusion. In this paper we would like to try to show the conclusion by numerical examples. The figure 3 shows  $\Phi_{M_0, M}(x, 2.0)$  for the parameters  $N = 2, L = 5, M_0 = 0$  for the parameters  $M = 0$  to 4  $M = L - 1$  and  $L = 4$  to 7. And the Figure 5 also shows the  $\Phi_{M_0, M}(x, y)$ . For the parameters  $N = 2, L = 5, M_0 = 0, M = 4$ . From these figures for the case

$$N = 2, M_0 = 0, M = L - 1 \quad (40)$$

we can see that  $\Phi_{M_0, M}(x, y)$  is depends only to  $x - y$ . Wavelet is symmetric function of  $x - y$ . And for  $M = L - 1$  the peak happens with minimum resolution  $1/2^L$ . That is, we can get

$$\Phi_{M_0, M}(x, y) \approx \begin{cases} 2^{L+1} & (y = x) \\ -2^{L-1} & (y = x \pm \frac{1}{2^L}) \\ 0 & (y \neq x \text{ \& } y \neq x \pm \frac{1}{2^L}). \end{cases} \quad (41)$$

The above results are derived for  $N = 2$ . Also we may get same conclusion for other  $N$ . By the summation by Cesaro we can get Fejer kernel. The Fejer kernel becomes positive real function therefore it is expected that phenomenon never happens. But in this paper we would like to use Gibbs phenomenon positively. Figure 2 shows the value of

$$\delta f_M(x) = f_M(x) - f(x) \quad (42)$$

$$L_{\max} = \max_{j < j_*} \left| f_M\left(\frac{j}{2^L}\right) - f\left(\frac{j}{2^L}\right) \right| \quad (43)$$

$$R_{\max} = \max_{j \geq j_*} \left| f_M\left(\frac{j}{2^L}\right) - f\left(\frac{j}{2^L}\right) \right|. \quad (44)$$

And it conditions sharp peaks at the discontinuous points  $a$  and the edges, where  $j_a = 2^L a$ . At this point which coordinate is given

by the multiplier of  $\frac{1}{2^L}$ ,  $L_{\max}, R_{\max}$  are given by

$$\begin{aligned} L_{\max} &= R_{\max} \\ &= \{(2^{L+1} - 2^{L-1}) \cdot g(a+0) - 2^{L-1} \cdot g(a-0)\} \cdot 2^{-L} - g(a+0) \\ &= \frac{1}{2} \{g(a+0) - g(a-0)\} \end{aligned} \quad (45)$$

Where function  $g(x)$  is assumed the right continuity  $g(a+0) = g(a)$ . Especially for  $f(x)$  in (10) we get  $L_{\max} = R_{\max} \approx 0.5$ . For other case of parameter we can also evaluate them by the calculation  $L_{\max}, R_{\max}$  directly. Table 1 and 2 show the  $L_{\max}, R_{\max}$  for several  $a, L$ . They show that  $L_{\max} = R_{\max}$  and they do not depend on the values  $a, L$ . Table 3 shows the dependence on the Daubechies parameter  $N$ . From the Table 3 we can see that  $R$  and  $L$  increase when  $N$  increases. But it is not monotone. By the way Haar function [13], [15] corresponds to the case of  $N = 1$  of Daubechies Wavelet. That is, the Daubechies Wavelet is an extension of Haar function. If the Haar function shows Gibbs phenomenon, then the very good situation is realized in a views point of the fast calculation. But the Haar functions are itself discontinuous and their finite expansion converges to its original function uniformly at  $\frac{j}{2^L} (0 \leq j < 2^L)$ .

Thus the Gibbs phenomenon never happens in the Haar function expansion. Therefore we can detect the grows of the missed observations in signal by using the finite Wavelet expansion without any prediction procedure.

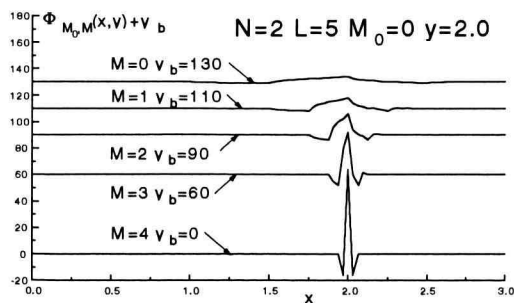


Fig.3 Dependency of  $\Phi_{M_0,M}(x,y)$  for  $M$

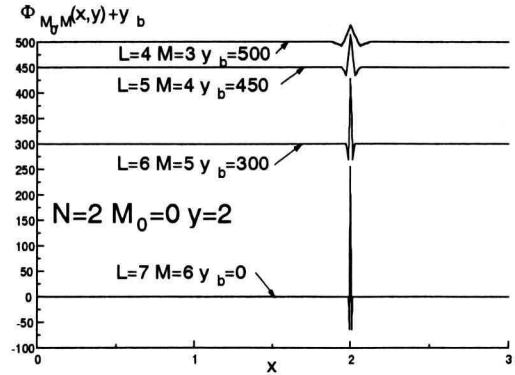


Fig.4 Dependency of  $\Phi_{M_0,M}(x,y)$  for  $L$

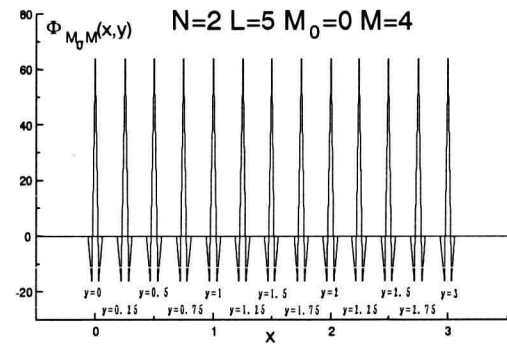


Fig.5 Dependency of  $\Phi_{M_0,M}(x,y)$  for  $y$

Table 2 Dependency for  $a$  ( $M_0 = 0, M = L - 1$ )

$N$	$L$	$a$	$L_{\max}$	$R_{\max}$
6	6	-4	0.02871401	0.02871403
6	6	-2	0.02871401	0.02871403
6	6	0	0.02871401	0.02871403
6	6	2	0.02871401	0.02871403
6	6	4	0.02871401	0.02871402

Table 3 Dependency for  $L$  ( $M_0 = 0, M = L - 1$ )

$N$	$L$	$a$	$L_{\max}$	$R_{\max}$
6	1	5	0.02871399	0.02871402
6	2	5	0.02871399	0.02871402
6	4	5	0.02871400	0.02871402
6	6	5	0.02871401	0.02871403

Table 4 Dependency for  $N$  ( $M_0 = 0, M = L - 1$ )

$N$	$L$	$a$	$L_{\max}$	$R_{\max}$
1	8	0.5	0.00000000	0.00000000
2	7	1	0.49834992	0.49834992
3	7	2	0.40628830	0.40629092
4	7	3	0.02724871	0.02724894
5	6	4	0.13956685	0.13956687
5	6	5	0.02871401	0.02871403
7	5	6	0.13806889	0.13806889
8	5	7	0.02913826	0.02913827
9	5	8	0.10514027	0.10514029
10	5	9	0.04227589	0.04227592

## 5. Detection of normal and missing states

In this section it is shown that we can detect the transition points between normal and missing states of signal by using Gibbs phenomenon generated in the finite Daubechies Wavelet expansion. Let us consider a function  $\xi(t)$ , which varies with time  $t$ . Let  $s(t)$  be the true original signal without any missing defects. And  $\rho(t)$  is a state of the observed signal  $\xi(t)$  given by

$$\rho(t) = \begin{cases} 0 & : \text{Normal State} \\ 1 & : \text{Missed State} \end{cases} \quad (46)$$

Usually we can express the observed signal  $\xi(t)$  as

$$\xi(t) = (1 - \rho(t))s(t) + \rho(t)\gamma(\rho(t), 0, \sigma_L^2) \quad (47)$$

Where  $\rho(t)$  is a stochastic process which takes a value 0 and 1. Also let  $\gamma(\rho(t), 0, \sigma_L^2)$  be a random variable, which varies the value only at the instance when  $\rho(t)$  transfers to 1 from 0 and has mean 0 and variance  $\sigma_L^2$ . This variable holds a constant value while observed signal keep the missing state. And the constant values are different on every missing states. Of course we don't know  $\rho(t)$ ,  $s(t)$  and  $\sigma_L^2$  exactly. But we are able to have the samples of the observations signal  $\xi(t)$ . Where it is assumed that the original  $s(t)$  is a band limited signal. Thus the angular frequency  $\omega$  which we can treat in the digital processing is limited

$$|\omega| \leq \omega_F = \frac{\pi}{T_s} \quad (48)$$

Where  $T_s$  is sampling period of the system. And we define the sampled values

$$\xi_k = \xi(kT_s), \quad s_k = s(kT_s), \quad \rho_k = \rho(kT_s) \quad (k \in \mathbf{Z}) \quad (49)$$

In this section for present time  $k$ , we examine to expect  $\rho_{k-d_1}$  where  $d_1$  is integer

$$d_1 > 0 \quad (50)$$

and at first let  $\xi_{k-N_H}, \xi_{k-N_H+1}, \dots, \xi_k$  be sampled signal at the time present  $k$  to the  $N_H$  past.

For the convention we put them as follows.

$$f_\ell = \xi_{k-N_H+\ell} \quad (\ell = 0, 1, \dots, N_H) \quad (51)$$

For some adequate  $f_0, f_1, \dots, f_{N_H}$ , we calculate the following spectrum of them

$$C_{M_0, q} = \frac{\sqrt{2^{M_0}}}{2^L} \sum_{\ell=0}^{N_H} f_\ell \cdot \varphi(2^{M_0} \frac{\ell}{2^L} - q) \quad (q \in \mathbf{Z}) \quad (52)$$

$$D_{p, q} = \frac{\sqrt{2^p}}{2^L} \sum_{\ell=0}^{N_H} f_\ell \cdot \psi(2^p \frac{\ell}{2^L} - q) \quad (p, q \in \mathbf{Z}). \quad (53)$$

Form (13) we get the finite expansion of  $f_\ell$  ( $\ell = 0, 1, \dots, N_H$ ) by

$$f_{M, \ell} = \sum_{n=nh_0}^{nh_0} C_{M_0, n} \sqrt{2^{M_0}} \phi(2^{M_0} \frac{\ell}{2^L} - n) + \sum_{m=M_0}^M \sum_{n=nh(m)}^{nh(m)} D_{m, n} \sqrt{2^m} \psi(2^m \frac{\ell}{2^L} - n). \quad (54)$$

The error by the Gibbs phenomenon is defined by

$$\Delta f_{M, \ell} = |f_{M, \ell} - f_\ell| \quad (\ell = 0, 1, \dots, N_H). \quad (55)$$

For adequate  $\delta > 0$  we define

$$\hat{\rho}_k = \begin{cases} 1 & \Delta f_{M, \ell} > \delta |f_\ell| \\ 0 & \Delta f_{M, \ell} \leq \delta |f_\ell| \end{cases} \quad (\text{at } \ell = N_H - d_1). \quad (56)$$

This means that as  $\hat{\rho}_k = 1$  only at the points where  $f_{N_H-d_1}$ , that is,  $\xi_{k-d_1}$  changes from normal state to missed one or from missed state to normal one. Therefore  $\hat{\rho}_k$  detects the transition points after delay time  $d_1$ . Where  $d_1$  is positive and used for avoiding the discontinuity at the left and right sides which yields on the finite interval sampling  $f_0, f_1, \dots, f_{N_H}$  as (52). This state is shown in Figure 2. For  $N = 2, M_0 = 0, M = L - 1, d_1 \geq 2$  is enough by (41). It is obvious that the

detection . method does not contain any prediction. The selection of delta is depending on the  $L_{\max}, R_{\max}$  at the discontinuous points of signal. For  $N = 2, M_0 = 0, M = L - 1$  the formula (45) is satisfied, thus  $\delta < 0.5$  (actually  $\delta \approx 0.2$ ) is preferable. For  $N$  the evaluation of Dirichlet kernel is necessary we detection method with the condition [A] can present.

## 6. Discrimination of normal and missed state

In this section we treat the discrimination of the normal and missed states after the detection of their transitions. If we get

$$\hat{\rho}_{k-1} = 1, \quad \hat{\rho}_k = 0 \quad (57)$$

after the missed state at the time  $k$  and furthermore after  $d_2$  (where  $d_2$  is integer) of the Gibbs phenomenon over, we try to calculate the observed variance

$$\sigma_k^2 = \frac{1}{d_2 + 1} \sum_{j=0}^{d_2} (\xi_{k+j} - \frac{1}{d_2 + 1} \sum_{l=0}^{d_2} \xi_{k+l})^2 \quad (58)$$

for the samples signal

$$\xi_k, \xi_{k+1}, \dots, \xi_{k+d_2}. \quad (59)$$

If observed signal is involved in the missed state, it does not vary. Therefore small positive  $\varepsilon$ ,  $\sigma_k^2 < \varepsilon$  is hold. Then we set

$$\mu_k = \begin{cases} 1 & \sigma_k^2 < \varepsilon \\ 0 & \sigma_k^2 \geq \varepsilon. \end{cases} \quad (60)$$

That is,  $\mu_k$  shows the missed states at time  $k$ . In the missed state there is no observation noise,  $\varepsilon$  is set in order to satisfy the variance  $\sigma_{Noise}^2$  of observation signal is discussed later. Then  $d_2$  is set smaller enough than the mean of interval (let it be  $\lambda_L T_s$ ) for example we put as  $d_2 \leq \lambda_L / 5$ , where

$$d = d_1 + d_2. \quad (61)$$

In order to estimate  $\mu_k$  we need to get the observed data till time  $k + d$ . Therefore the estimate  $\hat{\mu}_k$  at time  $k$  becomes

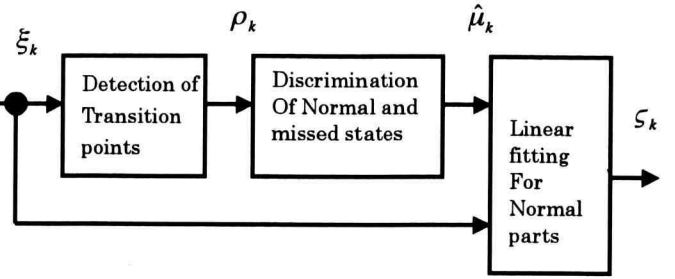


Fig.6 Block diagram of the proposed filter

$$\hat{\mu}_k = \mu_{k-d}. \quad (62)$$

That is, we can estimate the state whether the signal is normal or missing at the time delay  $d$  from present. It satisfies the condition [B].

## 7. Filtering

We can detect the missing states of the signal.

Next we choose  $R$  points

$$j_1 < j_2 < \dots < j_R \leq k - d \\ \hat{\mu}_{j_r+d} = 0 \quad (r = 1, 2, \dots, R) \quad (63)$$

in the normal state as possible as it is near  $j \leq k - d$  at present  $k$ . For the convention we set

$$\eta_r = \xi_{j_r} \quad (r = 1, 2, \dots, R) \quad (64)$$

Next the line is fitted to the points ( $j_r$  is greater than  $k - d$ )

$$\hat{\eta}(j) = a \cdot j + b \quad (65)$$

for ( $j_r, \eta_r$ ) ( $r = 1, 2, \dots, R$ ) by the least squared method.

$$\sum_{r=1}^R \{a \cdot j_r + b - \eta_r\}^2 \rightarrow \text{Min} \quad (66)$$

From (66) we can get the normal equation. And the solution of it, that is, the estimate  $\hat{a}_k, \hat{b}_k$  of  $a, b$  are given by

$$\begin{bmatrix} \hat{a}_k \\ \hat{b}_k \end{bmatrix} = \begin{bmatrix} \sum_{r=1}^R j_r^2 & \sum_{r=1}^R j_r \\ \sum_{r=1}^R j_r & R \end{bmatrix}^{-1} \begin{bmatrix} \sum_{r=1}^R j_r \cdot \eta_r \\ \sum_{r=1}^R \eta_r \end{bmatrix} \quad (67)$$

Our destination  $\zeta_k$  is expressed by

$$\zeta_k = \hat{a}_k \cdot (k - d) + \hat{b}_k. \quad (68)$$

$\zeta_k$  is the estimate of the original signals  $s_{k-d}$ , which reduces the additive noise.

$\hat{a}_k, \hat{b}_k$  are not constant but variants adaptively on  $k$ . In this paper the signal varies slowly. Thus

$1/R$  has to be arranged so as to be greater than the highest frequency  $f_{upper}$  of signal. It is preferable to choose the value of  $R$  less than  $1/10$  of  $1/f_{upper}$ , where  $f_{upper}$  is the highest frequency of significant parts of signal. By the procedure discussed above we are able to construct a filter which satisfies the conditions [A],[B] and [C]. The structure of the proposed filter is shown in the Figure 6.

## 8. Numerical examples

The proposed filter discussed by the above sections does not depend on the detail structure  $\rho(t)$  and  $\gamma(\rho(t), 0, \sigma_L^2)$ . For the numerical discussion we would like to set the values as follows. The statistical distribution of  $\rho(t)$ , the length of 0 that is normal states and 1 that is missed state change independently

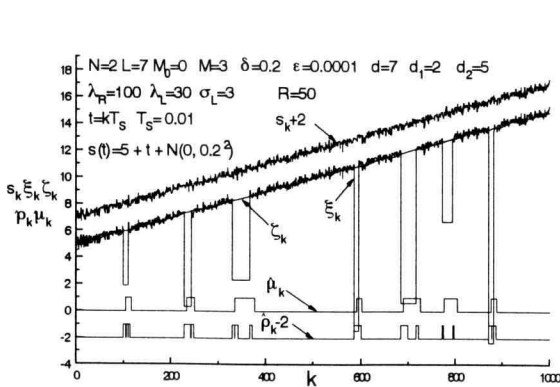


Fig.7 Processing for line

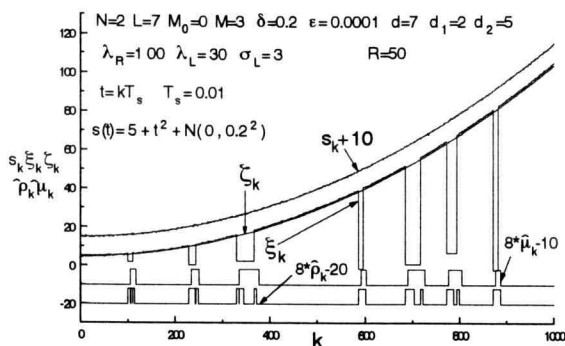


Fig.8 Processing for quadratic curve

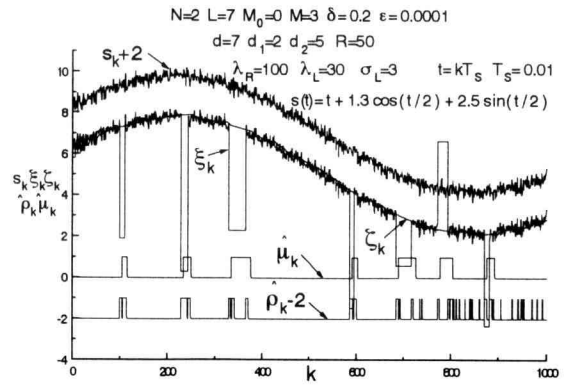


Fig.9 Processing for almost periodic function

on the exponential distribution with means

$\lambda_R T_s$  ( $\lambda_R$  is an integer) and  $\lambda_L T_s$  ( $\lambda_L$  is an integer) respectively. Let  $N$  be a normal distribution with mean and variance For the original signal we assume

$$s_k = \alpha + \beta \cdot k + \gamma \cdot k^2 + \sum_{n=1}^{N_s} \{A_n \cos(\theta_n k) + B_n \sin(\theta_n k)\} + N(0, \sigma_{Noise}^2). \quad (69)$$

The parameters  $\delta, \epsilon, d_1, d_2, R$  are set based on the discussion of section 4-6. Figure 7 Figure 8 and Figure 9 show the outputs of the proposed filter for the ramp, quadratic and almost periodic function of time respectively. The parameters used for these simulation are presented in their figures. They show the smooth compensation of the missed states in the observation signal. And the estimated signals are delayed  $d$  by first  $d$  from the original signals.

## 9. Conclusion

It is pointed out that the difference of the finite Daubechies Wavelet expansion and the observed signal shows sharp peaks at the discontinuous points of it. This phenomenon is equivalent to that occurred in the Fourier series expansion Using this phenomenon we proposed a detection method without any prediction for the missed observation data. By

this detection method we proposed a filter with noise reduction and constant delay properties. As this proposed filter is nonlinear, we can not treat it as frequency characteristics, but we have to discuss the optimal design procedure of its parameters for given  $\lambda_R, \lambda_L, \sigma_L^2, \sigma^2$ . But in this paper there is only the structure of the filter. Therefore we should study the optimal design of the parameters and the case of the signal folding caused by the dynamic range of the transmission scheme of signal.

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