On indices of primitive n-semigroups with an even number n

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Abstract

We introduce the notion of index of a numerical semigroup. Using this index we classify the primitive numerical semigroups starting with an even number and investigate whether such a numerical semigroup is non-Weierstrass.

Key Words: Non-singular curve, Primitive numerical semigroup

§1. Introduction.

In this paper, $\mathbf{N}$ will denote the additive semigroup of non-negative integers. A subsemigroup H of $\mathbf{N}$ is called a *numerical semigroup* if its complement $\mathbf{N}\backslash H$ in $\mathbf{N}$ is finite. The number $\sharp(\mathbf{N}\backslash H) = g(H)$ is called the *genus* of H. We set

$$L(H) = \mathbf{N}\backslash H = \{l_1 < l_2 < \cdots < l_{g(H)}\}.$$

Let C be a complete non-singular irreducible algebraic curve of genus g over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. For any point P of C, $H(P)$ denotes the set of integers which are pole orders at P of regular functions on $C\backslash\{P\}$. Then $H(P)$ is a numerical semigroup of genus g. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. Hurwitiz's original question was whether any numerical semigroup is Weierstrass[1)]. This was a long-standing problem. Buchweitz finally showed that not every numerical semigroup is Weierstrass[2)] : In fact, if H is Weierstrass, then we must have $\sharp L_m(H) \leq (2m-1)(g(H)-1)$ for each $m \geq 2$ where $L_m(H)$ denotes the set of all sums of m elements of $L(H)$, because the k-vector space of m-fold regular differentials on a curve of genus g has dimension $(2m-1)(g-1)$. He constructed numerical semigroups H which satisfy $\sharp L_m(H) > (2m-1)(g(H)-1)$ for some $m \geq 2$. For this reason we say that a numerical semigroup H is said to be *Buchweitz* if it satisfies $\sharp L_m(H) > (2m-1)(g(H)-1)$ for some $m \geq 2$. We are interested in investigating whether a given numerical semigroup is Buchweitz. For that purpose we define the following notions: A numerical semigroup H is called an *n-semigroup* if the minimum of positive integers in H is n. In this case the standard basis $S(H)$ for H is the set $\{n, s_1, \ldots, s_{n-1}\}$ where $s_i = \mathrm{Min}\{h \in H | h \equiv i \bmod n\}$ for each i. Let

$$s(H) = \mathrm{Min}\{s_i + s_{n-i} | i = 1, \ldots, \left[\frac{n}{2}\right]\}.$$

We denote by $\mathrm{Ind}(H)$ the non-negative integer

$$\frac{1}{n}\sum_{i=1}^{[\frac{n}{2}]}(s_i + s_{n-i} - s(H)),$$

which is called the *index* of the n-semigroup H. In this paper, we are devoted to the primitive n-semigroups of a given index with an even number n, where a *primitive* n-semigroup H means that the largest integer which does not belong to H is less than $2n$. Our aim is to investigate whether such a numerical semigroup is Buchweitz.

§2. Classification of the primitive n-semigroups with an even number n by their indices.

Let n be an integer ≥ 2. In this section let H be a primitive n-semigroup. Then we have

$$\{1, \ldots, n-1\} \subseteq L(H) \subseteq \{1, \ldots, n-1, n+1, \ldots, 2n-1\},$$

which implies that $n-1 \leq g(H) \leq 2(n-1)$. In the primitive case we get the following results for the values of $s(H)$ and $\mathrm{Ind}(H)$.

Lemma 2.1 *We have* $s(H) = 3n$ *or* $4n$ *or* $5n$.

Proof. Since $n < s_i < 3n$ for all i, we get our result. □

Proposition 2.2 *The integer* $\mathrm{Ind}(H)$ *is between* 0 *and* $2([n/2]-1)$, *where* $[r]$ *denotes the integral part of* r.

Proof. The number $\mathrm{Ind}(H)$ takes the maximum if there exists i with $1 \leq i \leq [n/2]$ such that $n+i, n+n-i \in S(H)$ and $n+j \notin S(H)$ for each j which is distinct from i and $n-i$. Hence we get

$$\mathrm{Ind}(H) \leq \frac{1}{n}\left(\left[\frac{n}{2}\right]-1\right)(5n-3n) = 2\left(\left[\frac{n}{2}\right]-1\right). \quad \square$$

From now on we assume that n is an even number. We set $\delta = 1$ (resp. 2) if $s_{n/2} = n + n/2$ (resp. $2n + n/2$). There is an important and useful relation between $g(H)$, $s(H)$ and $\mathrm{Ind}(H)$.

Proposition 2.3 *We have*

$$g(H) = \frac{n}{2}\left(\frac{s(H)}{n} - 1\right) + \mathrm{Ind}(H) - \delta.$$

Proof. Since we have

$$g(H) = \sum_{i=1}^{\frac{n}{2}}\left(\frac{s_i + s_{n-i}}{n} - 1\right) - \delta \text{ and } \mathrm{Ind}(H) = \frac{1}{n}\sum_{i=1}^{\frac{n}{2}}(s_i + s_{n-i} - s(H)),$$

we get the above relation. □

For the indices of the primitive n-semigroups we get the following results :

Theorem 2.4 (0) $0 \leq \mathrm{Ind}(H) \leq n-2$.

(1) *If* $s(H) = 5n$, *then* H *is uniquely detemined. In fact,* $L(H) = \{1, \ldots, n-1, n+1, \ldots, 2n-1\}$. *In this case,* $\mathrm{Ind}(H) = 0$.

(2) *If* $s(H) = 4n$, *then* $0 \leq \mathrm{Ind}(H) \leq \dfrac{n-2}{2}$.

(3) *If* $s(H) = 3n$, *then* $\mathrm{Ind}(H) = g(H) - (n - \delta)$.

Proof. The inequalities (0) follow from Proposition 2.2. The statement (1) is obvious. Let $s(H) = 4n$. Then we have

$$\mathrm{Ind}(H) = \frac{1}{n}\sum_{i=1}^{\frac{n}{2}}(s_i + s_{n-i} - 4n) \leq \frac{n-2}{2}.$$

The relation (3) follows directly from Proposition 2.3. □

§3. Non-Weierstrass n-semigroups with an even number n which are Buchweitz.

We assume that n is an even number ≥ 6, because we know that all 4-semigroups are Weierstrass[3]. In this section let H be a primitive n-semigroup. We denote by $w(H)$ the integer $\sum_{i=1}^{g(H)}(l_i - i)$, which is called the *weight* of H. First we show that if $s(H)$ and $\mathrm{Ind}(H)$ take some border values, then H is Weierstrass.

Proposition 3.1 *An n-semigroup H is Weierstrass if it satisfies one of the following conditions :*
(1) $s(H) = 5n$.
(2) $s(H) = 4n$ *and* $\mathrm{Ind}(H) = (n-2)/2$.
(3) $s(H) = 3n$ *and* $\mathrm{Ind}(H) = n - 2$.
(4) $s(H) = 3n$ *and* $\mathrm{Ind}(H) = 1$.
(5) $s(H) = 3n$ *and* $\mathrm{Ind}(H) = 0$.

Proof. (1) If $s(H) = 5n$, then $w(H) = n - 1 \leq g(H) - 2 = 2n - 4$. Hence H is Weierstrass[4].
(2) Let $s(H) = 4n$ and $\mathrm{Ind}(H) = (n-2)/2$. Then there is i with $1 \leq i \leq n-1$ such that $s_i = n + i$ and $s_j = 2n + j$ for all $j \neq i$. Hence

$$L(H) = \{1, \ldots, n-1, n+1, \ldots, n+i-1, n+i+1, \ldots, n+n-1\},$$

which implies that $w(H) = 2n - 3 - i \leq g(H) - 1 = 2n - 4$. Hence H is Weierstrass[4),5)].
(3) Let $s(H) = 3n$ and $\mathrm{Ind}(H) = n - 2$. Then there exists i with $1 \leq i \leq n/2$ such that

$$L(H) = \{1, \ldots, n-1, n+1, \ldots, 2n-1\}\backslash\{n+i, n+n-i\},$$

which implies that $w(H) = 2n - 6 \leq g(H) - 2$. Therefore H is Weierstrass.
(4) Let $s(H) = 3n$ and $\mathrm{Ind}(H) = 1$. Then there is i with $1 \leq i \leq n-1$ such that $s_i = 2n + i$, $s_{n-i} = n+n-i$ and for all $j \neq i, n-i$, $s_j = n+j$, $s_{n-j} = n+n-j$. Hence $L(H) = \{1, \ldots, n-1, n+i\}$, which implies that $w(H) \leq g(H) - 1$. Thus H is Weierstrass.
(5) Let $s(H) = 3n$ and $\mathrm{Ind}(H) = 0$. Then $L(H) = \{1, \ldots, n-1\}$, which implies that H is Weierstrass.
□

We do not know whether all n-semigroups H with $s(H) = 4n$ are Weierstrass, but we get the following result :

Theorem 3.2 *All n-semigroups H with $s(H) = 4n$ are non-Buchweitz.*

Proof. In view of $L(H) \subseteq \{1, \ldots, n-1, n+1, \ldots, 2n-1\}$, we have $L_m(H) \subseteq \{m, \ldots, m(2n-1)\}$ for all $m \geq 2$, which implies that $\sharp L_m(H) \leq m(2n-1) - (m-1)$. In view of $s(H) = 4n$, we must have

$s_{n/2} = 2n + n/2$. By Proposition 2.3 we get $g(H) \geq \dfrac{3n-4}{2}$, which implies that

$$(2m-1)(g(H)-1) \geq m(3n-6) - \frac{3(n-2)}{2}.$$

Hence we obtain

$$(2m-1)(g(H)-1) - \sharp L_m(H) \geq (n-4)(m-2) + \frac{n-12}{2}.$$

Thus if $n \geq 12$ (resp. $n = 8, 10$ and $m \geq 3$, resp. $n = 6$ and $m \geq 4$), then we get $(2m-1)(g(H)-1) \geq \sharp L_m(H)$. By Propositions 2.3 and 3.1 (2) we may assume that $g(H) \leq 2n-3$. If $n = 8$ or 10, then $g(H) \leq 17$, which implies that $3(g(H)-1) \geq \sharp L_2(H)$[6]. If $n = 6$ and $m \geq 4$, then $g(H) \leq 9$, which implies that H is Weierstrass[7),8)]. □

In the remainder of this section we will investigate whether H with $s(H) = 3n$ is non-Buchweitz. In the case where $\mathrm{Ind}(H) = 2$ we see the following result :

Proposition 3.3 *If H is an n-semigroup of index 2 with $s(H) = 3n$, then it is non-Buchweitz.*

Proof. We have $L(H) = \{1, \ldots, n-1, l_n, l_{n+1}\}$ or $\{1, \ldots, n-1, n+n/2\}$. Hence the statement follows from Theorem 1.6 in 6). □

In the case where H has heigher index we get the following results :

Theorem 3.4 *Let H be an n-semigroup with $s(H) = 3n$.*
(1) *If* $\mathrm{Ind}(H) \geq \left\lceil \dfrac{n+2}{3} \right\rceil + \delta$, *then H is non-Buchweitz.*
(2) *If* $\dfrac{n}{5} + \delta \leq \mathrm{Ind}(H) \leq \dfrac{n}{3} + \delta$, *then we have*

$$\sharp L_m(H) \leq (2m-1)(g(H)-1) \text{ for all } m \geq 3.$$

Proof. By the similar way to the proof of Theorem 3.2 we have

$$(2m-1)(g(H)-1) - \sharp L_m(H) \geq (2m-1)(\mathrm{Ind}(H) - \delta) - n.$$

Hence we get our results. □

Corollary 3.5 *If $n \leq 12$, then all primitive n-semigroups H are non-Buchweitz.*

Proof. By Proposition 3.1, Theorems 3.2 and 3.4 (1), and Proposition 3.3 we only consider the case where $s(H) = 3n$ and $3 \leq \mathrm{Ind}(H) \leq \left\lceil \dfrac{n+2}{3} \right\rceil + \delta - 1$. Let $n = 6$. It suffices to only consider the case where $\mathrm{Ind}(H) = 3$ and $s_3 = 15$. Then $g(H) = 7$, which implies that it is Weierstrass[7)]. Let $n = 8$. It suffices to consider the cases where "$\mathrm{Ind}(H) = 3$ and $s_4 = 12, 20$" or "$\mathrm{Ind}(H) = 4$ and $s_4 = 20$". If $\mathrm{Ind}(H) = 3$ and $s_4 = 20$, then $g(H) = 9$, which implies that it is Weierstrass[8)]. If "$\mathrm{Ind}(H) = 3$ and $s_4 = 12$" or "$\mathrm{Ind}(H) = 4$ and $s_4 = 20$", then by Theorem 3.4 (2) it suffices to check that $\sharp L_2(H) \leq 3g(H) - 3$. In view of $g(H) = 10$ the inequality holds[6)]. Let $n = 10$. First we consider the case $s_5 = 15$. From Theorem 3.4 (1) and (2) it suffices to check that $\sharp L_2(H) \leq 3g(H) - 3$ when $\mathrm{Ind}(H) = 3$ or 4. Since $g(H) = 12$ or 13, we get the desired result[6)]. Next we consider the case $s_5 = 25$. If $\mathrm{Ind}(H) = 4$ or 5, by Theorem 3.4 and 6) H is non-Buchweitz. If $\mathrm{Ind}(H) = 3$, then there is i with $1 \leq i \leq 9$ and $i \neq 5$ such that $L(H) = \{1, \ldots, 9, 10+i, 15\}$. By Theorem 1.6 in 6) H is non-Buchweitz. Let $n = 12$. First we consider the case $s_6 = 18$. It suffices to only investigate the case $\mathrm{Ind}(H) = 3$.

Then $L(H) = \{1, \dots, 11, 12+i, 12+j, 12+k\}$, where i, j and k are distinct positive integers which are less than 12. By Theorem 4.3 in 6) we have $\sharp L_2(H) \le 3g(H) - 3$ and $\sharp L_3(H) \le 5g(H) - 5$. Using the inequality in the proof of Theorem 3.4 we get $\sharp L_m(H) \le (2m-1)(g(H)-1)$ for all $m \ge 4$. Next we consider the case $s_6 = 30$. In this case we must investigate the cases $\mathrm{Ind}(H) = 3, 4$. If $\mathrm{Ind}(H) = 4$, then $L(H) = \{1, \dots, 11, 18, 12+i, 12+j\}$, where i and j are distinct positive integers which are less than 12 and which are different from 6. By the same method as above we can show that H is non-Buchweitz. If $\mathrm{Ind}(H) = 3$, then $L(H) = \{1, \dots, 11, 18, 12+i\}$, where i is a positive integer which is less than 12 and which is different from 6. It follows from Theorem 1.6 in 6) that H is non-Buchweitz. □

Using the results in 6) we get the following :

Example 3.6 For any $n \ge 18$ we denote by $H(n)$ the n-semigroup with

$$L(H(n)) = \{1, \dots, n-1, 2n-8, 2n-3, 2n-2\}.$$

Then $s(H(n)) = 3n$ and $\mathrm{Ind}(H) = 3$. By Theorem 4.7 in 6) if $n \ge 44$, then $\sharp L_7(H(n)) > 13g(H(n)) - 13$. Thus $H(n)$ is Buchweitz.

Proposition 3.7 *Let d be an integer with $4 \le d \le \left[\frac{n-1}{4}\right] + 2$. If $n \ge 14$, then there exists an n-semigroup H of index d with $s(H) = 3n$ which is Buchweitz.*

Proof. We set $\alpha(H) = (\alpha_0, \alpha_1, \dots, \alpha_{g(H)-1})$ where $\alpha_i = l_{i+1} - i - 1$ for each $i = 0, 1, \dots, g(H)-1$. Let $l \ge 3$ and $n - 1 \ge 4l$. We denote by H the n-semigroup with

$$\alpha(H) = (0^{n-1}, n-1-2l, n-1-2l+1, \dots, n-1-2l+l-2, n-1-l, n-1-l).$$

Then by Proposition 2.2 (1) in 6) we have $\sharp L_2(H) = 3g(H) - 2$, which implies that H is Buchweitz. Since $n+3$ and $n+n-3$ belong to H, we have $s(H) = 3n$, which implies that $\mathrm{Ind}(H) = l + \delta$ by Proposition 2.4 (3). We note that in the case $l = (n-2)/4$ we have $s_{n/2} = 2n + n/2$, otherwise $s_{n/2} = n + n/2$. Let $n \equiv 0 \bmod 4$. Then $s_{n/2} = n + n/2$, which implies that $\mathrm{Ind}(H) = l + 1$ with $3 \le l \le (n-4)/4$. Thus we have $4 \le \mathrm{Ind}(H) \le \left[\frac{n-1}{4}\right] + 1$. Let $n \equiv 2 \bmod 4$. In the case $l = (n-2)/4$ we have $\mathrm{Ind}(H) = (n-2)/4 + 2$, otherwise $\mathrm{Ind}(H) = l + 1$. Thus we get $4 \le \mathrm{Ind}(H) \le \left[\frac{n-1}{4}\right]$ or $\mathrm{Ind}(H) = \left[\frac{n-1}{4}\right] + 2$.

Moreover, we consider other n-semigroups H' as follows : For any integer l with $l \ge 3$ and $n - 1 \ge 4l$, let

$$\alpha(H') = (0^{n-1}, n-1-2l, n-1-2l, n-1-2l+1, \dots, n-1-2l+l-1, n-1-2l+l-1).$$

Then $\sharp L_2(H') = 3g(H') - 3 + l - 2 \ge 3g(H') - 2$, which implies that H' is Buchweitz. Let $n \equiv 0 \bmod 4$. Then $s_{n/2} = n + n/2$, which implies that $\mathrm{Ind}(H') = l + 2$ with $3 \le l \le (n-4)/4$. Thus we have $5 \le \mathrm{Ind}(H') \le \left[\frac{n-1}{4}\right] + 2$. Let $n \equiv 2 \bmod 4$. In the case $l = (n-2)/4$ we have $\mathrm{Ind}(H') = (n-2)/4 + 3$, otherwise $\mathrm{Ind}(H') = l + 2$. Thus we get $5 \le \mathrm{Ind}(H') \le \left[\frac{n-1}{4}\right] + 1$ or $\mathrm{Ind}(H') = \left[\frac{n-1}{4}\right] + 3$. Hence we get our desired examples. □

In the remaining cases we get the following proposition :

Proposition 3.8 *For each d with $\left[\frac{n-1}{4}\right] + 3 \le d \le \left[\frac{n+2}{3}\right] + 1$ there exists an n-semigroup H*

of index d with $s(H)=3n$ which is Buchweitz.

Proof. We set $i=d-\left[\frac{n-1}{4}\right]-4$. Let $n\equiv 0 \bmod 4$. For any d with $\left[\frac{n-1}{4}\right]+4\le d\le\left[\frac{n+2}{3}\right]+1$ we denote by $H(n,d)$ the n-semigroup

$$L(H(n,d))=\{1,\dots,n-1,n+\frac{n}{2}-1,n+\frac{n}{2},n+\frac{n}{2}+2,\dots,$$

$$n+\frac{n}{2}+2i+2,n+\frac{n}{2}+2\cdot(i+2),\dots,n+\frac{n}{2}+2\cdot\frac{n-4}{4},2n-1\}.$$

Then we have $g(H(n,d))=n+d-2$ and $s(H(n,d))=3n$. In view of $s_{n/2}=2n+n/2$ it follows from Theorem 2.4 (3) that $\mathrm{Ind}(H(n,d))=d$. It is easy to check that $L_2(H(n,d))=\{2,\dots,4n-2\}$, which implies that $\sharp L_2(H(n,d))-(3g(H(n,d)-2)=n-3d+5\ge 0$. Thus $H(n,d)$ is a Buchweitz n-semigroup of index d. Next, we consider the case $d=\left[\frac{n-1}{4}\right]+3$. For any $n\ge 20$ we denote by $H(n,d)$ the n-semigroup

$$L(H(n,d))=\{1,\dots,n-1,n+\frac{n}{2}-1,n+\frac{n}{2}+1,n+\frac{n}{2}+2\cdot 1,\dots,n+\frac{n}{2}+2\cdot\frac{n-4}{4},2n-1\}.$$

Then we have $g(H(n,d))=n+d-1$ and $s(H(n,d))=3n$. In view of $s_{n/2}=n+n/2$ it follows from Theorem 2.4 (3) that $\mathrm{Ind}(H(n,d))=d$. Moreover, we have $L_2(H(n,d))=\{2,\dots,3n-2,3n,\dots,4n-2\}$, which implies that $\sharp L_2(H(n,d))-(3g(H(n,d)-2)=(n-20)/4\ge 0$. If $n=16$, then we denote by $H(16,6)$ the 16-semigroup

$$L(H(16,6))=\{1,\dots,15,20,22,27,28,30,31\}.$$

Then we have $g(H(16,6))=21$, $s(H)=48$ and $\mathrm{Ind}(H(16,6))=6$. Moreover, we have $L_2(H(16,6))=\{2,\dots,62\}$, which implies that $\sharp L_2(H(16,6))=3g(H(16,6))-2$.

Let $n\equiv 2 \bmod 4$. For any d with $\left[\frac{n-1}{4}\right]+4\le d\le\left[\frac{n+2}{3}\right]+1$ we denote by $H(n,d)$ the n-semigroup

$$L(H(n,d))=\{1,\dots,n-1,n+\frac{n}{2}-1,n+\frac{n}{2},n+\frac{n}{2}+2,\dots,$$

$$n+\frac{n}{2}+2i+2,n+\frac{n}{2}+2\cdot(i+1)+1,\dots,n+\frac{n}{2}+2\cdot\frac{n-6}{4}+1,2n-1\}.$$

Then we have $g(H(n,d))=n+d-2$, $s(H(n,d))=3n$ and $\mathrm{Ind}(H(n,d))=d$. Now we have $L_2(H(n,d))=\{2,\dots,4n-2\}$, which implies that $\sharp L_2(H(n,d))-(3g(H(n,d)-2)=n-3d+5\ge 0$. Moroeover, the proof of Proposition 3.7 gives one of the Buchweitz n-semigroups of index $\left[\frac{n-1}{4}\right]+3$. □

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