Remarks on Weierstrass pairs of ramification points on a bielliptic curve

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Abstract

We construct Weierstrass pairs of ramification points on a bielliptic curve, that is, a double covering of an elliptic curve.

Key Words: Weierstrass point, Weierstrass pair of points, bielliptic curve

§1. Introduction.

Let C be a complete nonsingular curve of genus $g \geq 2$ over an algebraically closed field of characteristic 0, which is called a *curve* in this paper, and $\mathbb{K}(C)$ the field of rational functions on C. Let P be a point of C. We define the Weierstrass semigroup $H(P)$ of the point P by

$$H(P) = \{\alpha \in \mathbb{N} | \text{ there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

where $\mathbb{N}$ denotes the additive semigroup of non-negative integers. The least positive integer in $H(P)$ is called the *first non-gap of P*. The point P is called a *Weierstrass point* if $\mathbb{N}\backslash H(P) \neq \{1, 2, \dots, g\}$, i.e., dim $H^0(C, \mathcal{O}_X(gP)) \geq 2$ where for any divisor D on C we set

$$H^0(C, \mathcal{O}_C(D)) = \{f \in \mathbb{K}(C) | \mathrm{div}(f) \geq -D\}.$$

Let P and Q be distinct points of C. We call the pair (P, Q) a *Weierstrass pair of points* if there exist positive integers α and β with $\alpha + \beta = g - 1$ such that dim $H^0(C, \mathcal{O}_C(\alpha P + \beta Q)) \geq 2$. We define the Weierstrass semigroup $H(P, Q)$ of the pair (P, Q) of points P and Q by

$$H(P, Q) = \{(\alpha, \beta) \in \mathbb{N} \times \mathbb{N} | \text{ there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P + \beta Q\}.$$

Arbarello, Cornalba, Griffiths and Harris[1] introduced the notions of a Weierstrass pair and the Weierstrass semigroup of a pair of points on a curve and they investigated their properties. Kim[2] and Homma[3] developed their idea. In the case of a hyperelliptic curve C, the semigroup $H(P, Q)$ is determined explicitly by Kim[2]. In the case of a curve C of genus 3, the semigroup $H(P, Q)$ is also determined by Kim and Komeda[4]. The semigroup $H(P, Q)$ of the pair (P, Q) of points whose first non-gaps are three is determined completely by Kim and Komeda[5]. Moreover, Kim and Komeda determine the semigroup $H(P, Q)$ of ramification points P and Q on a cyclic covering of the projective line $\mathbb{P}^1$ of degree 4[6] (resp. prime degree $p \geq 5$[7]).

In this paper, we construct Weierstrass pairs of ramification points on a bielliptic curve using the method of the construction of a bielliptic curve which is due to Park[8].

§2. Park's construction of a bielliptic curve.

In this section we shall review the semigroup $H(P)$ of a ramification point P on a bielliptic curve and the construction of a bielliptic curve of genus g from an elliptic curve according to Park's method[8].

Remark 2.1. Let C be a bielliptic curve of genus $g \geq 6$.
i) There exists a unique double covering map from C to an elliptic cuve E. Let $\pi : C \longrightarrow E$ be the unique double covering.
ii) If P is a ramification point of π, then the semigroup $H(P)$ is either $< 4, 6, 2g-3 >$ or $< 4, 6, 2g-1, 2g+1 >$ where for positive intergers $a_1, \ldots, a_n$ we denote by $< a_1, \ldots, a_n >$ the semigroup generated by $a_1, \ldots, a_n$.

Definition 2.2. A ramification point P on a bielliptic curve C of genus $g \geq 6$ is said to be *of heavy type* (resp. *of light type*) if $H(P) =< 4, 6, 2g-3 >$ (resp. $< 4, 6, 2g-1, 2g+1 >$).

We reveiw Park's method[8]. Let E be the elliptic curve defined by a lattice $\Lambda = \gamma_1\mathbb{Z} + \gamma_2\mathbb{Z}$, that is, $E = \mathbb{C}/\Lambda$, where γ_1 and γ_2 are complex numbers which are linearly independent over $\mathbb{R}$. Then we have a one to one correspondence between the set $\Phi = \{t_1\gamma_1 + t_2\gamma_2 : 0 \leq t_1, t_2 < 1\}$ and the points of E. Let

$$\Delta_1 = \left\{ \frac{m_1}{g-1}\gamma_1 + \frac{m_2}{g-1}\gamma_2 : m_1, m_2 = 0, 1, \ldots, g-2 \right\} \backslash \{0\}$$

and

$$\Delta_2 = \left\{ \frac{n_1}{g-2}\gamma_1 + \frac{n_2}{g-2}\gamma_2 : n_1, n_2 = 0, 1, \ldots, g-3 \right\} \backslash \{0\}.$$

Then the subsets Δ_1 and Δ_2 of Φ are disjoint. The cardinality of Δ_1 (resp. Δ_2) is $g^2 - 2g$ (resp. $g^2 - 4g + 3$). We define a subset $\tilde{\Delta}_1$ of $\Delta_1 \times \Delta_1$ and a subset $\tilde{\Delta}_2$ of $\Delta_2 \times \Delta_2$ as follows:

$$\tilde{\Delta}_1 = \left\{ \left(\frac{m_1}{g-1}\gamma_1, \frac{(g-1)-m_1}{g-1}\gamma_1 \right) \in \Delta_1 \times \Delta_1 : m_1 = 1, 2, \ldots, \left[\frac{g-2}{2}\right] \right\}$$
$$\cup \left\{ \left(\frac{m_2}{g-1}\gamma_2, \frac{(g-1)-m_2}{g-1}\gamma_2 \right) \in \Delta_1 \times \Delta_1 : m_2 = 1, 2, \ldots, \left[\frac{g-2}{2}\right] \right\}$$
$$\cup \left\{ \left(\frac{m_3}{g-1}\gamma_1 + \frac{m_3}{g-1}\gamma_2, \frac{(g-1)-m_3}{g-1}\gamma_1 + \frac{(g-1)-m_3}{g-1}\gamma_2 \right) \in \Delta_1 \times \Delta_1 : m_3 = 1, 2, \ldots, \left[\frac{g-2}{2}\right] \right\}$$
$$\cup \left\{ \left(\frac{1}{g-1}\gamma_1 + \frac{2}{g-1}\gamma_2, \frac{g-2}{g-1}\gamma_1 + \frac{g-3}{g-1}\gamma_2 \right) \right\},$$

$$\tilde{\Delta}_2 = \left\{ \left(\frac{n_1}{g-2}\gamma_1, \frac{(g-2)-n_1}{g-2}\gamma_1 \right) \in \Delta_2 \times \Delta_2 : n_1 = 1, 2, \ldots, \left[\frac{g-3}{2}\right] \right\}$$
$$\cup \left\{ \left(\frac{n_2}{g-2}\gamma_2, \frac{(g-2)-n_2}{g-2}\gamma_2 \right) \in \Delta_2 \times \Delta_2 : n_2 = 1, 2, \ldots, \left[\frac{g-3}{2}\right] \right\}$$
$$\cup \left\{ \left(\frac{n_3}{g-2}\gamma_1 + \frac{n_3}{g-2}\gamma_2, \frac{(g-2)-n_3}{g-2}\gamma_1 + \frac{(g-2)-n_3}{g-2}\gamma_2 \right) \in \Delta_2 \times \Delta_2 : n_3 = 1, 2, \ldots, \left[\frac{g-3}{2}\right] \right\}$$
$$\cup \left\{ \left(\frac{1}{g-2}\gamma_1 + \frac{2}{g-2}\gamma_2, \frac{g-3}{g-2}\gamma_1 + \frac{g-4}{g-2}\gamma_2 \right) \right\}$$
$$\cup \left\{ \left(\frac{2}{g-2}\gamma_1 + \frac{1}{g-2}\gamma_2, \frac{g-4}{g-2}\gamma_1 + \frac{g-3}{g-2}\gamma_2 \right) \right\}.$$

Note that $\sharp\tilde{\Delta}_1 \geq g$ and $\sharp\tilde{\Delta}_2 \geq g-1$. Using the above sets Park[8] consructed bielliptic curves of genus g with s ramification points of heavy type for $s \leq 2g-2$ and $s \neq 2, 2g-3$. For example, let s be an even integer with $4 \leq s \leq 2g-2$. We set

$$\tilde{\Delta}'_1 = \tilde{\Delta}_1 \backslash \left\{ \left(\frac{1}{g-1}\gamma_1, \frac{g-2}{g-1}\gamma_1 \right), \left(\frac{1}{g-1}\gamma_1 + \frac{1}{g-1}\gamma_2, \frac{g-2}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2 \right), \left(\frac{1}{g-1}\gamma_2, \frac{g-2}{g-1}\gamma_2 \right) \right\}.$$

Choose $\frac{s-4}{2}$ elements $(x_1, x_2), (x_3, x_4), \dots, (x_{s-5}, x_{s-4})$ in $\tilde{\Delta}_1'$. Set

$$x_0 = 0, x_{s-3} = \frac{1}{g-1}\gamma_1, x_{s-2} = \frac{g-2}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2 \text{ and } x_{s-1} = \frac{1}{g-1}\gamma_2.$$

Let $t = 2g-2-s$. Choose $\frac{t}{2}$ elements $(y_1, y_2), (y_3, y_4), \dots, (y_{t-1}, x_t)$ in $\tilde{\Delta}_2$. Then it follows that

$$\sum_{i=1}^{s-1} x_i + \sum_{j=1}^{t} y_j - (2g-3)x_0 \equiv 0 \bmod \Lambda,$$

$$(g-1)(x_0 - x_i) \equiv 0 \bmod \Lambda \text{ for } i = 1, \dots, s-1 \text{ and } (g-2)(x_0 - y_j) \equiv \bmod \Lambda \text{ for } j = 1, \dots, t.$$

Let P_i (resp. Q_j) be the point on E corresponding to x_i (resp. y_j). Then there exist f, f_i and h_j in $\mathbb{K}(E)$ such that

$$(f) = P_1 + \dots + P_{s-1} + Q_1 + \dots + Q_t - (2g-3)P_0,$$

$$(f_i) = (g-1)(P_0 - P_i) \text{ for } i = 1, \dots, s-1 \text{ and } (h_j) = (g-2)(P_0 - Q_j) \text{ for } j = 1, \dots, t.$$

Let C be the curve with function field $\mathbb{K}(E)(f^{\frac{1}{2}})$. By Riemann-Hurwitz formula C is a bielliptic curve of genus g. If $\tilde{P}_i$ (resp. $\tilde{Q}_j$) be the preimage of P_i (resp. Q_j), then it is of heavy type (resp. light type).

§3. Weierstrass pairs of ramification points on a bielliptic curve.

In this section we construct Weierstrass pairs (P, Q) of ramification points on a bielliptic curve with $mP \sim mQ$ and $nP \not\sim nQ$ for any $n < m$ when the integer m satisfies certain conditions, where for any divisors D and D', $D \sim D'$ means that D and D' are linearly equivalent. First we note the following:

Remark 3.1 Let C be a bielliptic curve of genus $g \geq 6$. If P and Q are ramification points on C, then (P, Q) is a Weierstrass pair of points.

Proof. Let $\pi : C \longrightarrow E$ be the double covering of the elliptic curve E. If we set $P' = \pi(P)$ and $Q' = \pi(Q)$, then there is $f \in \mathbb{K}(E)$ such that $(f)_\infty = P' + Q'$. Hence we get $(\pi^*(f))_\infty = 2P + 2Q$, because P and Q are ramification points. Thus, we have dim $H^0(C, \mathcal{O}_X(\alpha P + \beta Q)) \geq 2$ for integers α and β which are at least 2, which implies that the pair (P, Q) is Weierstrass. □

By the above Remark it suffices to find ramification points P and Q on a bielliptic curve of genus g with $mP \sim mQ$ and $nP \not\sim nQ$ for any $n < m$ when the integer m satisfies certain conditions.

I. *The case where P and Q are of heavy type.* The divisors $(2g-2)P$ and $(2g-2)Q$ are canonical, because P and Q are of heavy type. Hence we get $(2g-2)P \sim (2g-2)Q$. Thus, we may assume that $m \leq 2g-2$ and that $2g-2$ is divisible by m. Moreover, m must be even. Suppose that m were odd. We have $m \in < 4, 6, 2g-3 >$, because P is of heavy type. Hence, $m = 2g-3$, which implies that $P \sim Q$. This is a contradiction. Now we fix an even integer $m = 2m'$ with $2 \leq m' \leq g-1$ such that $g-1$ is divisible by m'. Let E be the elliptic curve in Section 2. The notations in Section 2 will be used. First, let $2 < m' < g-1$. Then $g \geq 7$. In this case we get $\left(\frac{1}{m'}\gamma_1, \frac{m'-1}{m'}\gamma_1\right) \in \tilde{\Delta}_1'$. We can choose $g-3$ elements $(x_1, x_2), \dots, (x_{2g-7}, x_{2g-6})$ in $\tilde{\Delta}_1'$ such that $x_1 = \frac{1}{m'}\gamma_1$ and $x_2 = \frac{m'-1}{m'}\gamma_1$. Set $x_0 = 0, x_{2g-5} = \frac{1}{g-1}\gamma_1, x_{2g-4} = \frac{g-2}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2$ and $x_{2g-3} = \frac{1}{g-1}\gamma_2$. Then, it follows that

$$x_1 + x_2 + \sum_{i=3}^{2g-6} x_i + \frac{\gamma_1}{g-1} + \left(\frac{g-2}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2\right) + \frac{1}{g-1}\gamma_2 - (2g-3)x_0 \equiv 0 \bmod \Lambda$$

$$\text{and } (g-1)(x_0 - x_i) \equiv 0 \bmod \Lambda \text{ for } i = 1, \dots, 2g-3.$$

Moreover, $m'(x_0 - x_1) \equiv 0 \bmod \Lambda$ and $n'(x_0 - x_1) \not\equiv 0 \bmod \Lambda$ for $1 \le n' < m'$. Let P_i be the point on E corresponding to x_i for $i = 0, 1, \ldots, 2g-3$. Then there exist f and f_1 in $\mathbb{K}(E)$ such that

$$(f) = \sum_{i=1}^{2g-3} P_i - (2g-3)P_0 \quad \text{and} \quad (f_1) = m'(P_0 - P_1).$$

Moreover, for any $1 \le n' < m'$ there is no $h \in \mathbb{K}(E)$ such that $(h) = n'(P_0 - P_1)$. Let C be the curve with function field $\mathbb{K}(E)(f^{\frac{1}{2}})$. Let $\pi : C \longrightarrow E$ be the double covering. We denote the preimage of P_i by $\tilde{P}_i$. Since $H(\tilde{P}_0) \ni 2g-3$, the ramification point $\tilde{P}_0$ is of heavy type. Moreover, for any $i = 1, \ldots, 2g-3$ the ramification point $\tilde{P}_i$ is also of heavy type, because $(2g-2)\tilde{P}_i \sim (2g-2)\tilde{P}_0$. If $n\tilde{P}_0 \sim n\tilde{P}_1$ with $n < 2m' = m \le 2g-2$, then n must be even. But this contradicts the minimality of m'. Therefore, if we set $P = \tilde{P}_0$ and $Q = \tilde{P}_1$, then the pair (P, Q) is the desired one. Second, let $m' = g-1$. In this case, choose $g-3$ elements $(x_1, x_2), \ldots, (x_{2g-7}, x_{2g-6})$ in $\tilde{\Delta}'_1$, and set $x_0 = 0, x_{2g-5} = \dfrac{1}{g-1}\gamma_1, x_{2g-4} = \dfrac{g-2}{g-1}\gamma_1 + \dfrac{g-2}{g-1}\gamma_2$ and $x_{2g-3} = \dfrac{1}{g-1}\gamma_2$. If we set $P = \tilde{P}_0$ and $Q = \tilde{P}_{2g-5}$, then the pair (P, Q) is the desired one. Lastly, let $m' = 2$. Set

$$\tilde{\Delta}''_1 = \tilde{\Delta}_1 \backslash \left\{ \left(\frac{1}{g-1}\gamma_1, \frac{g-2}{g-1}\gamma_1 \right), \left(\frac{\frac{g-3}{2}}{g-1}\gamma_1, \frac{\frac{g+1}{2}}{g-1}\gamma_1 \right) \right\}.$$

Then $\sharp\tilde{\Delta}''_1 \ge g-2$. Choose $g-3$ elements $(x_1, x_2), \ldots, (x_{2g-7}, x_{2g-6})$ in $\tilde{\Delta}''_1$. Set $x_0 = 0, x_{2g-5} = \dfrac{1}{g-1}\gamma_1, x_{2g-4} = \dfrac{\frac{g-3}{2}}{g-1}$ and $x_{2g-3} = \dfrac{\frac{g-1}{2}}{g-1}\gamma_1 = \dfrac{1}{2}\gamma_1$. Then, it follows that

$$\sum_{i=1}^{2g-6} x_i + x_{2g-5} + x_{2g-4} + x_{2g-3} - (2g-3)x_0 \equiv 0 \bmod \Lambda.$$

In the similar way to the case $2 < m' < g-1$ if we set $P = \tilde{P}_0$ and $Q = \tilde{P}_{2g-3}$, the pair (P, Q) is the desired one. □

II. *The case where P is of heavy type and Q is of light type with $m \le 2g-2$.* The divisor $(2g-2)P$ is canonical and $(2g-2)Q$ is not canonical. Hence we get $(2g-2)P \not\sim (2g-2)Q$. Thus, we may assume that $2g-2$ is not divisible by m. Moreover, m must be even, because of $m \le 2g-2$. Now we fix an even integer $m = 2m'$ with $2 \le m' \le g-1$ such that $g-1$ is not divisible by m'. Choose $g-4$ elements $(x_1, x_2), \ldots, (x_{2g-9}, x_{2g-8})$ in $\tilde{\Delta}'_1$. Set

$$x_0 = 0, x_{2g-7} = \frac{1}{g-1}\gamma_1, x_{2g-6} = \frac{g-2}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2,$$

$$x_{2g-5} = \frac{1}{g-1}\gamma_2, x_{2g-4} = \frac{m'-1}{m'}\gamma_1 \quad \text{and} \quad x_{2g-3} = \frac{1}{m'}\gamma_1.$$

Since $g-1$ is not divisible by m', $x_0, x_1, \ldots, x_{2g-4}, x_{2g-3}$ are distinct. Then, it follows that

$$\sum_{i=1}^{2g-3} x_i - (2g-3)x_0 \equiv 0 \bmod \Lambda \quad \text{and} \quad m'(x_0 - x_{2g-3}) \equiv 0 \bmod \Lambda.$$

Moreover, for any $1 \le n' < m'$ we have $n'(x_0 - x_{2g-3}) \not\equiv 0 \bmod \Lambda$. Let P_i be the point on E corresponding to x_i for $i = 0, 1, \ldots, 2g-3$. Then there is $f \in \mathbb{K}(E)$ such that $(f) = \sum_{i=1}^{2g-3} P_i - (2g-3)P_0$. Let the notation be as in I. If we set $P = \tilde{P}_0$ and $Q = \tilde{P}_{2g-3}$, then the pair (P, Q) satisfies $mP \sim mQ$ and $nP \not\sim nQ$ for any $n < m$. □

III. *The case where P and Q are of light type with $m \le 2g-2$ and $m|(2g-2)$.* Since $m \le 2g-2$, m must be even. Now we fix an even integer $m = 2m'$ with $2 \le m' \le g-1$ such that $g-1$ is divisible by

m'. First, let $m' \neq 2, g-1$. We set

$$\tilde{\Delta}_1''' = \tilde{\Delta}_1 \backslash \left\{ \left(\frac{1}{g-1}\gamma_1, \frac{g-2}{g-1}\gamma_1 \right), \left(\frac{1}{g-1}\gamma_1 + \frac{2}{g-1}\gamma_2, \frac{g-2}{g-1}\gamma_1 + \frac{g-3}{g-1}\gamma_2 \right), \left(\frac{3}{g-1}\gamma_1, \frac{g-4}{g-1}\gamma_1 \right) \right\}.$$

We note that Park used this set in his paper[8]. Choose $g-4$ elements $(y_1, y_2), (y_3, y_4), \dots, (y_{2g-9}, y_{2g-8})$ in $\tilde{\Delta}_1'''$ which include $\left(\frac{1}{m'}\gamma_1, \frac{g-1-q}{g-1}\gamma_1 \right)$ where we set $q = \frac{g-1}{m'}$, for example, let $y_1 = \frac{1}{m'}\gamma_1$. Moreover, set

$$y_0 = 0, y_{2g-7} = \frac{1}{g-1}\gamma_1 + \frac{2}{g-1}\gamma_2, y_{2g-6} = \frac{1}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2,$$

$$y_{2g-5} = \frac{g-3}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2, y_{2g-4} = \frac{g-4}{g-1}\gamma_1 \quad \text{and} \quad y_{2g-3} = \frac{1}{g-1}\gamma_1.$$

Then we obtain

$$\sum_{i=1}^{2g-4} y_i + 3y_{2g-3} - (2g-1)y_0 \equiv 0 \bmod \Lambda.$$

Let Q_i be the point on E corresponding to y_i for $i = 0, 1, \dots, 2g-3$. Hence, there exists $f \in \mathbb{K}(E)$ such that

$$(f) = \sum_{i=1}^{2g-4} Q_i + 3Q_{2g-3} - (2g-1)Q_0.$$

Let the notation be as in I. If we set $P = \tilde{Q}_0$ and $Q = \tilde{Q}_1$, then the pair (P, Q) is the desired one. Let $m' = g-1$. If we set $P = \tilde{Q}_0$ and $Q = \tilde{Q}_{2g-3}$, then the pair (P, Q) is the desired one. Lastly, let $m' = 2$. We set

$$\tilde{\Delta}_1^{(4)} = \tilde{\Delta}_1''' \backslash \left\{ \left(\frac{2}{g-1}\gamma_1, \frac{g-3}{g-1}\gamma_1 \right), \left(\frac{\frac{g-3}{2}}{g-1}\gamma_1, \frac{\frac{g+1}{2}}{g-1}\gamma_1 \right) \right\}.$$

Choose $g-5$ elements $(y_1, y_2), (y_3, y_4), \dots, (y_{2g-11}, y_{2g-10})$ in $\tilde{\Delta}_1^{(4)}$. Set

$$y_0 = 0, y_{2g-9} = \frac{1}{2}\gamma_1, y_{2g-8} = \frac{\frac{g-3}{2}}{g-1}\gamma_1, y_{2g-7} = \frac{1}{g-1}\gamma_1 + \frac{2}{g-1}\gamma_2,$$

$$y_{2g-6} = \frac{1}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2, y_{2g-5} = \frac{g-3}{g-1}\gamma_1 + \frac{g-2}{g-1}\gamma_2, y_{2g-4} = \frac{g-3}{g-1}\gamma_1 \quad \text{and} \quad y_{2g-3} = \frac{1}{g-1}\gamma_1.$$

Then, we obtain

$$\sum_{i=1}^{2g-4} y_i + 3y_{2g-3} - (2g-1)y_0 \equiv 0 \bmod \Lambda.$$

Let Q_i be the point on E corresponding to y_i for $i = 0, 1, \dots, 2g-3$. Hence, there exists $f \in \mathbb{K}(E)$ such that

$$(f) = \sum_{i=1}^{2g-4} Q_i + 3Q_{2g-3} - (2g-1)Q_0.$$

If we set $P = \tilde{Q}_0$ and $Q = \tilde{Q}_{2g-9}$, then the pair (P, Q) satisfies the required condition. □

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