A generalization of a non-symmetric numerical semigroup generated by three elements

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Abstract

We investigate the property of a non-symmetric numerical semigroup generated by three elements. We consider the numerical semigroups generated by at least 4 elements with the property which is said to be neat. The concept of a neat semigroup is a generalization of that of a non-symmetric semigroup generated by three elements. Moreover, the notions of "quasi-toric type" and "toric type" are introduced to neat semigroups. We show that every neat semigroup generated by 4 elements is of quasi-toric type. We give several semigroups of toric type.

Key Words: Non-symmetric numerical semigroup, Relation module for a semigroup, Saturated semigroup

§1. Introduction.

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}$ is called a *numerical semigroup of genus g* if its complement $\mathbb{N}\backslash H$ is a finite set consisting of g elements. We denote the genus of H by $g(H)$. We define

$$c(H) = \mathrm{Min}\{c \in \mathbb{N} | c + \mathbb{N} \subset H\},$$

which is called the *conductor* of H. Then the inequality $c(H) \leq 2g(H)$ holds. The numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$. In this paper, we concern ourselves with non-symmetric numerical semigroups generated by three elements. Firstly, we invetsigate the property of these semigroups. Second, we consider numerical semigroups generated by at least four elements with the same property as the non-symmetric semigroups generated by three elements has, which are called neat numerical semigroups. We introduce a new notion into neat numerical semigroups, which is said to be of quasi-toric type. A numerical semigroup of quasi-toric type means that we can associate an affine toric variety to its relations. We give several numerical semigroups of quasi-toric type. Especially, we prove that any neat numerical semigroup generated by four elements is of quasi-toric type. Moreover, we consider a 1-neat numerical semigroup, which means that it is a neat numerical semigroup with one more condition. We can show that a 1-neat numerical semigroup generated by four elements is of toric type, which implies that it is Weierstrass. Roughly speaking, a numerical semigroup of quasi-toric type is said to be of toric type if the defining ideal of its associated affine toric variety describes that of the semigroup ring of H.

§2. Non-symmetric numerical semigroups generated by three elements.

First we review Herzog's result[1)] on non-symmetric numerical semigroups generated by three elements.

Definition 2.1. For a numerical semigroup H we denote by $M(H)$ the minimal set of generators for H. Let $M(H) = \{a_1, a_2, \ldots, a_n\}$. For each $i = 1, 2, \ldots, n$ we set

$$\alpha_i = \mathrm{Min}\{\alpha | \alpha a_i \in < a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n >\}$$

where for any non-negative integers $b_1, \ldots, b_m$ the set $< b_1, \ldots, b_m >$ means the semigroup generated by $b_1, \ldots, b_m$.

Example 2.2. i) Let $H =< 4, 5, 6 >$. Then $g(H) = 4$ and $c(H) = 8$. Hence H is symmetric. If we set $a_1 = 4$, $a_2 = 5$ and $a_3 = 6$, then $\alpha_1 = 3$, $\alpha_2 = 2$ and $\alpha_3 = 2$.

ii) Let $H =< 4, 5, 7 >$. Then $g(H) = 4$ and $c(H) = 7$. Hence H is non-symmetric. If we set $a_1 = 4$, $a_2 = 5$ and $a_3 = 7$, then $\alpha_1 = 3$, $\alpha_2 = 3$ and $\alpha_3 = 2$.

The following fact is due to Herzog.

Remark 2.3. Let H be a non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. We have the following relations:

$$\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3,\ \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3,\ \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2$$

where $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$ and $0 < \alpha_{ij} < \alpha_j$, all i, j [1)]. In this case α_{ij}'s are uniquely determined.

Example 2.4. Let the notation be as in Example 2.2 ii). Then we have the folloing relations.

$$3a_1 = a_2 + a_3,\ 3a_2 = 2a_1 + a_3,\ 2a_3 = a_1 + 2a_2.$$

Moreover, we get the following result on the coefficients of the relations:

Lemma 2.5. *Let the notation be as in Remark 2.3. We have* $\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = \nu a_3$ *for some positive integer* ν.

Proof. Consider

$$\begin{cases} \alpha_1 a_1 - \alpha_{12} a_2 = \alpha_{13} a_3. \\ -\alpha_{21} a_1 + \alpha_2 a_2 = \alpha_{23} a_3 \end{cases}$$

We get

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = \alpha_1 \alpha_2 - \alpha_{12}\alpha_{21} = \alpha_{21}\alpha_{32} + \alpha_{31}(\alpha_{12} + \alpha_{32}) > 0.$$

Therefore we have

$$a_1 = a_3 \begin{vmatrix} \alpha_{13} & -\alpha_{12} \\ \alpha_{23} & \alpha_2 \end{vmatrix} \Big/ \begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} \text{ and } a_2 = a_3 \begin{vmatrix} \alpha_1 & \alpha_{13} \\ -\alpha_{21} & \alpha_{23} \end{vmatrix} \Big/ \begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix}.$$

Since $H =< a_1, a_2, a_3 >$ is a numerical semigroup, we have $(a_1, a_2, a_3) = 1$. Hence $\begin{vmatrix} \alpha_1 & \alpha_{13} \\ -\alpha_{21} & \alpha_{23} \end{vmatrix}$ must be divisible by a_3. □

Proposition 2.6. *Let the notation be as in Remark 2.3. Then we have* $\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = a_3.$

Proof. We set

$$L_1 = \{\alpha a_1 + \beta a_2 | 0 \leq \alpha \leq \alpha_{31} - 1, 0 \leq \beta \leq \alpha_2 - 1, (\alpha, \beta) \neq (0,0)\}$$

$$\text{and } L_2 = \{\gamma a_1 + \delta a_2 | \alpha_{31} \leq \gamma \leq \alpha_1 - 1, 0 \leq \delta \leq \alpha_{32} - 1\}.$$

Let $L = L_1 \cup L_2 \cup \{a_3\}$. We want to show that $l \not\equiv l'$ mod a_3 for distinct l and $l' \in L$. First, it is proved that $l \not\equiv 0$ mod a_3 for any $l \in L_1 \cup L_2$. In fact, assume that $l = \alpha a_1 + \beta a_2 \in L_1$ such that $l \equiv 0$ mod a_3. Then $\kappa a_3 = \alpha a_1 + \beta a_2$ with $\kappa \geq \alpha_3$. We get $(\kappa - \alpha_3)a_3 + (\alpha_{31} - \alpha)a_1 = (\beta - \alpha_{32})a_2$. Since $0 < \beta - \alpha_{32} < \alpha_2 - 1$, this is a contradiction. Thus, we get $l \not\equiv 0$ mod a_3 for $l \in L_1$. Similarly we obtain $l \not\equiv 0$ mod a_3 for $l \in L_2$. Second, we can show that $l \not\equiv l'$ mod a_3 for any distinct l and $l' \in L_1$. Let $l = \alpha a_1 + \beta a_2$ and $l' = \alpha' a_1 + \beta' a_2$. Assume that $l \equiv l'$ mod a_3. Then we may assume that $(\alpha - \alpha')a_1 + (\beta - \beta')a_2 = \kappa a_3$ with $\kappa \geq \alpha_3$. If $\alpha - \alpha' \geq 0$ and $\beta - \beta' \geq 0$, then we have

$$(\beta - \beta')a_2 = (\kappa - \alpha_3)a_3 + (\alpha_{31} - (\alpha - \alpha'))a_1 + \alpha_{32}a_2 \text{ with } 0 \leq \beta - \beta' \leq \alpha_2 - 1,$$

which is a contradiction. In the other cases we also get a contradiction. Similarly it is showed that $l \not\equiv l'$ mod a_3 for any distinct l and $l' \in L_2$. Lastly, let $l = \alpha a_1 + \beta a_2 \in L_1$ and $l' = \gamma a_1 + \delta a_2 \in L_2$. Assumee that $l \equiv l'$ mod a_3. If $l \geq l'$, then we have $(\alpha - \gamma)a_1 + (\beta - \delta)a_2 = \kappa a_3$ with $\kappa > 0$, which is a contradiction. If $l < l'$, then we have $(\gamma - \alpha)a_1 + (\delta - \beta)a_2 = \kappa a_3$ withh $\kappa > 0$, which implies that

$$(\gamma - \alpha)a_1 = (\kappa - \alpha_3)a_3 + \alpha_{31}a_1 + (\alpha_{32} - (\delta - \beta))a_2.$$

This is a contradiction. By the above we get

$$a_3 \geq \sharp L = \alpha_2\alpha_{31} - 1 + (\alpha_1 - \alpha_{31})\alpha_{32} + 1 = \alpha_2\alpha_{31} + \alpha_{21}\alpha_{32}.$$

It follows from Lemma 2.5 that

$$\alpha_2\alpha_{31} + \alpha_{21}\alpha_{32} = \alpha_1\alpha_2 - \alpha_{12}\alpha_{21} = \nu a_3 \geq a_3,$$

which implies that $\alpha_1\alpha_2 - \alpha_{12}\alpha_{21} = a_3$. □

Example 2.7. Let the notation be as in Example 2.2 ii). Then we have

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = 7 = a_3.$$

§3. Numerical semigroups of quasi-toric type.

First we generalize the notion of a non-symmetric numerical semigroups generated by three elements.

Definition 3.1. i) Let H be a numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$. A system of relations

$$\begin{cases} \alpha_1 a_1 = \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n a_n = \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1} \end{cases}$$

satisfying

$$\alpha_j = \alpha_{1j} + \cdots + \alpha_{j-1j} + \alpha_{j+1j} + \cdots + \alpha_{nj} \text{ for any } j = 1, \ldots, n \text{ and } 0 \leq \alpha_{ij} < \alpha_j \text{ for all } i, j$$

is said to be *neat*.

ii) A numerical semigroup H is said to be *neat* if it has a neat system of relations.

Example 3.2. i) Any non-symmetric numerical semigroup H with $\sharp M(H)=3$ is neat by Remark 2.3.
ii) Let H be a numerical semigroup with $M(H)=\{10,11,13,14\}$. We set $a_1=10$, $a_2=11$, $a_3=13$ and $a_4=14$. There is a unique neat system of relations

$$4a_1=2a_3+a_4,\ 3a_2=2a_1+a_3,\ 3a_3=a_2+2a_4 \text{ and } 3a_4=2a_1+2a_2,$$

which implies that H is neat.
iii) Let H be a numerical semigroup with $M(H)=\{20,24,25,31\}$. We set $a_1=20$, $a_2=24$, $a_3=25$ and $a_4=31$. We have a unique neat system of relations

$$4a_1=a_2+a_3+a_4,\ 4a_2=2a_1+a_3+a_4,\ 3a_3=a_1+a_2+a_4 \text{ and } 3a_4=a_1+2a_2+a_3,$$

which implies that H is neat.
iv) For any integer $n\geq 5$ let H_n be a numerical semigroup with

$$M(H_n)=\{a_1=n,a_2=n+1,a_3=2n+3,a_4=2n+4,\ldots,a_{n-1}=2n+n-1\}.$$

Then we have a neat system of relations

$$\alpha_1a_1=4a_1=a_2+a_{n-1},\ \alpha_2a_2=3a_2=a_1+a_3,\ \alpha_3a_3=2a_3=2a_2+a_4,$$

$$\alpha_ia_i=2a_i=a_{i-1}+a_{i+1}\ (4\leq i\leq n-2),\ \alpha_{n-1}a_{n-1}=2a_{n-1}=3a_1+a_{n-2}.$$

Hence H_n is a neat numerical semigroup.

Here we study a relation module for a numereical semigroup in order to introduce a new notion into its syetem of relations.

Definition 3.3. Let H be a numerical semigroup with $M(H)=\{a_1,\ldots,a_n\}$. The $\mathbb{Z}-$module $R=\{(r_1,r_2,\ldots,r_n)\in\mathbb{Z}^n|\sum_{i=1}^{m}r_ia_i=0\}$ is called a *relation module for H*, which is uniquely determined if the numbering of the elements of $M(H)$ is fixed.

Lemma 3.4. *Let the notation be as in Definition 3.3. Then a relation module for H is a free $\mathbb{Z}-$module of rank $n-1$.*

Proof. Let $\rho^*:\mathbb{N}^n\longrightarrow H$ be a homomorphism of semigroups sending e_i to a_i where e_i denotes the vector whose $i-$th component is 1 and $j-$th component is 0 for $j\neq i$. We set $\rho=\{(v,v')|\rho^*(v)=\rho^*(v')\}$, which is an equivalence relation on $\mathbb{N}^n$. Then $\mathbb{N}^n/\rho\cong H$, which implies that $\mathbb{Z}^n/R_\rho(H)\cong\mathbb{Z}$ [1)] where we set $R_\rho(H)=\{v-v'|(v,v')\in\rho\}$. We have an exact sequence

$$0\longrightarrow R_\rho(H)\longrightarrow\mathbb{Z}^n\longrightarrow\mathbb{Z}\longrightarrow 0$$

of free $\mathbb{Z}-$modules, which implies that

$$\text{rank } R_\rho(H)=\text{rank }\mathbb{Z}^n-\text{rank }\mathbb{Z}=n-1.$$

Since $R_\rho(H)$ is the relation module for H, we get our desired result. □

We need the following definition when we associate an affine toric variety with a semigroup.

Definition 3.5. Let S be a subsemigroup of $\mathbb{Z}^n$. We say that S is *saturated* if the condition $nr\in S$, where n is a positive integer and r an element of $\mathbb{Z}^n$, implies that $r\in S$.

Using Lemma 3.4 we can introduce the following definition:

Definition 3.6. Consider an order on the set $I = \{(i,j) | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$, which is fixed. Let H be a neat numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$. Take a neat system of relations

$$\begin{cases} \alpha_{21}a_1 + \cdots + \alpha_{n1}a_1 = \alpha_1 a_1 = \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{1n-1}a_{n-1} + \cdots + \alpha_{n-2n-1}a_{n-1} + \alpha_{nn-1}a_{n-1} = \alpha_{n-1}a_{n-1} = \alpha_{n-11}a_1 + \cdots + \alpha_{n-1n-2}a_{n-2} + \alpha_{n-1n}a_n \\ \alpha_{1n}a_n + \cdots + \alpha_{n-1n}a_n = \alpha_n a_n = \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1} \end{cases}$$

Assume that

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0.$$

We set $N = \sharp\{(i,j) | \alpha_{ij} \neq 0\} - (n-1)$. We associate the vector in $\mathbb{Z}^N$ with $\alpha_{ij}a_j$ by induction on i which means the i-th relation in the neat system of relations. Let i be fixed. We denote by (k_i, l_i) the maximum of the set

$$L_i = \Big(\{(j,i) | \alpha_{ji} \neq 0\} \cup \{(i,j) | \alpha_{ij} \neq 0\}\Big) \cap \Big(I \setminus \cup_{p=1}^{i-1} L_p\Big).$$

We number successively the elements (i,j) of the set L_i by $\sigma(i,j)$ in the given order if $(i,j) \neq (k_i, l_i)$. We associate the vector $b_{\sigma(i,j)} = e_{\sigma(i,j)}$ with $\alpha_{ij}a_j$ if $(i,j) \neq (k_i, l_i)$. For $\alpha_{k_i l_i}a_{l_i}$ we consider

$$\alpha_{k_i l_i}a_{l_i} = \cdots\cdots \pm \alpha_{pq}a_p \cdots\cdots$$

from the $i-$ th relation in the neat system. Using the relation we can associate the vector b_{N+i} with $\alpha_{k_i l_i}a_{l_i}$, because we already have associated some vector with $\alpha_{pq}a_p$. Thus, we can construct the subsemigroup $S = < b_1, \ldots, b_{N+n-1} >$ of $\mathbb{Z}^N$. The neat numerical semigroup H is said to be of *quasi-toric type* if the semigroup S is saturated.

The reason why H is said to be of quasi-toric type is that if the associated semigroup S is saturated then the affine scheme Spec $k[S]$ becomes an affine toric variety where k is an algebraically closed field.

Remark 3.7. Let the notation be as in Definition 3.6. Let the neat system of relations be fixed. Then the property of "quasi-toric type" does not depend on the choices of the numbering of the elements of $M(H)$ and the order on the set $I = \{(i,j) | 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$, because the two subsemigroups of $\mathbb{Z}^N$ are isomorphic through a suitable bijective correspondence of the set I.

Example 3.8. Let H be a non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. By Remark 2.3 we have the neat system of relations

$$\begin{cases} \alpha_1 a_1 = (\alpha_{21} + \alpha_{31})a_1 = \alpha_{12}a_2 + \alpha_{13}a_3. \\ \alpha_2 a_2 = (\alpha_{12} + \alpha_{32})a_2 = \alpha_{21}a_1 + \alpha_{23}a_3 \\ \alpha_3 a_3 = (\alpha_{13} + \alpha_{23})a_3 = \alpha_{31}a_1 + \alpha_{32}a_2 \end{cases}$$

We define the order on the set $\{(i,j) | i,j = 1,2,3 \text{ and } i \neq j\}$ as follows: $(i,j) \leq (i',j')$ if "$j < j'$" or "$j = j'$, $i \leq i'$". Then the associated subsemigroup S of $\mathbb{Z}^4$ is generated by $b_1 = e_1$, $b_2 = e_2$, $b_3 = e_3$, $b_4 = e_4$, $b_5 = (1,1,-1,0)$ and $b_6 = (-1,0,1,1)$. To show that S is saturated it suffices to check that $\sum_{i=1}^{6} \mathbb{R}_+ b_i \cap \mathbb{Z}^4 \subseteq S$ where $\mathbb{R}_+$ denotes the set of non-negative real numbers. In fact, let $z = \sum_{i=1}^{6} \lambda_i b_i \in \mathbb{Z}^4$ with $\lambda_i \in \mathbb{R}_+$. We may assume that $0 \leq \lambda_i < 1$ for each i. Then it is easy to see that $z \in S$. Thus, S is saturated, which implies that H is of quasi-toric type.

We get several examples of quasi-toric type in the case where $\sharp M(H) = n$.

Proposition 3.9. *Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ such that it has a neat system of relations*

$$\begin{cases} \alpha_1 a_1 = \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_n a_n = \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1} \end{cases}$$

with

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0.$$

Moreover, suppose that $\alpha_{ij} > 0$ for all i, j. Then H is of quasi-toric type.

Proof. We note that $\{(i,j) | \alpha_{ij} \neq 0\} = \{(i,j) | 1 \leq i \leq n, 1 \leq j \leq n \text{ and } i \neq j\} = I$, which implies that its cadinality is $n(n-1)$. We define the order on the set I as follows: $(i,j) \leq (i',j')$ if "$j < j'$" or "$j = j'$, $i \leq i'$". Then we get the associated subsemigroup

$$S = < b_1 = e_1, \dots, b_{(n-1)^2} = e_{(n-1)^2}, b_{(n-1)^2+1}, \dots, b_{n(n-1)} >$$

of $\mathbb{Z}^{(n-1)^2}$ through the method in Definition 3.6. Consider the $n(n-1)$ by $(n-1)^2$ matrix

$$(b_1 \cdots b_{(n-1)^2} b_{(n-1)^2+1} \cdots b_{n(n-1)}).$$

In the matrix -1 appears at most once in each column, from which it is easy to show that S is saturated. Therefore, H is of quasi-toric type. □

Example 3.10. Let H be the numerical semigroup with $M(H) = \{20, 24, 25, 31\}$ in Example 3.2 iii). By Proposition 3.9 it is of quasi-toric type.

We can generalize the above Proposition to the following two statements:

Proposition 3.11. *Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ as in Proposition 3.9. Suppose that there is some j_0 with $1 \leq j_0 \leq n$ such that $\alpha_{ij_0} > 0$ for all i. Then H is of quasi-toric type.*

Proof. Interchanging the numbering j_0 and n we may assume that $j_0 = n$. We introduce the order on the set I as follows: $(i,j) \leq (i',j')$ if "$j < j'$" or "$j = j'$, $i \leq i'$". Let i be fixed. Then -1 appears at the position of $\alpha_{ij}a_j$ for each $j \neq i$ in the vector corresponding to $\alpha_{in}a_n$. But 0 or 1 appears at the position of $\alpha_{ij}a_j$ for each $j \neq i$ in the vector corresponding to $\alpha_{kn}a_n$ for each $k \neq i$. It is easily seen that the associated semigroup S is saturated. Therefore, H is of quasi-toric type. □.

Proposition 3.12. *Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ as in Proposition 3.9. Suppose that there is some i_0 with $1 \leq i_0 \leq n$ such that $\alpha_{i_0j} > 0$ for all j. Then H is of quasi-toric type.*

Proof. Interchanging the numbering i_0 and n we may assume that $i_0 = n$. We introduce the order on the set I as follows: $(i,j) \leq (i',j')$ if "$i < i'$" or "$i = i'$, $j \leq j'$". Let j be fixed. Then -1 appears at the position of $\alpha_{ij}a_j$ for each $i \neq j$ in the vector corresponding to $\alpha_{nj}a_j$. But 0 or 1 appears at the position of $\alpha_{ij}a_j$ for each $i \neq j$ in the vector corresponding to $\alpha_{nk}a_k$ for each $k \neq j$. It is easily seen that the associated semigroup S is saturated. Therefore, H is of quasi-toric type. □.

Proposition 3.13. *Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ such that it* *has a neat system of relations*

$$\begin{cases} \alpha_1 a_1 = \alpha_{1n} a_n + \alpha_{12} a_2 \\ \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_i a_i = \alpha_{ii-1} a_{i-1} + \alpha_{ii+1} a_{i+1} \ (2 \leq i \leq n-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n a_n = \alpha_{nn-1} a_{n-1} + \alpha_{n1} a_1 \end{cases}$$

with

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0$$

where we set $\alpha_{ij} = 0$ if $\alpha_{ij} a_j$ does not appear in the system of relations. Then H is of quasi-toric type.

Proof. We introduce the order on the set I as follows: $(i,j) \leq (i',j')$ if "$j < j'$" or "$j = j'$, $i \geq i'$". Then we get the associated subsemigroup

$$S = < b_1 = e_1, \ldots, b_{n+1} = e_{n+1}, b_{n+2}, \ldots, b_{2n} >$$

of $\mathbb{Z}^{n+1}$ through the method in Definition 3.6 where

$$b_{n+2} = (1, 1, -1, 0, \ldots, 0) \text{ and } b_{n+j-1} = e_{-2,3,j} \text{ for } j = 4, \ldots, n+1.$$

Here $e_{-2,3,j}$ is the vector in $\mathbb{Z}^{n+1}$ whose second component is -1, third component is 1, j−th component is 1 and the other components are 0. Then S is saturated[2)]. Hence H is of quasi-toric type. □.

Example 3.14. For any integer $n \geq 5$ let H_n be a numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n+1, a_3 = 2n+3, a_4 = 2n+4, \ldots, a_{n-1} = 2n+n-1\}$$

in Example 3.2 iv). By Proposition 3.13 H_n is of quasi-toric type.

We can prove that a neat numerical semigroup is of quasi-toric type if it is genrated by four elements.

Theorem 3.15. *Let H be a neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$. Then it is of quasi-toric type.*

Proof. Let

$$\begin{cases} \alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3 + \alpha_{14} a_4 \\ \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 + \alpha_{24} a_4 \\ \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2 + \alpha_{34} a_4 \\ \alpha_4 a_4 = \alpha_{41} a_1 + \alpha_{42} a_2 + \alpha_{43} a_3 \end{cases}$$

be a unique neat system of relations for H. We note that

$$D = \begin{vmatrix} \alpha_1 & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_2 & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_3 \end{vmatrix} = \alpha_{41} \begin{vmatrix} -\alpha_{12} & -\alpha_{13} \\ \alpha_2 & -\alpha_{23} \end{vmatrix} - \alpha_{42} \begin{vmatrix} \alpha_1 & -\alpha_{13} \\ -\alpha_{21} & -\alpha_{23} \end{vmatrix} + \alpha_{43} \begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix}$$

$$= \alpha_{41}(\alpha_{12}\alpha_{23} + \alpha_2\alpha_{13}) + \alpha_{42}(\alpha_1\alpha_{23} + \alpha_{21}\alpha_{13}) + \alpha_{43}(\alpha_1(\alpha_{32} + \alpha_{42}) + (\alpha_{31} + \alpha_{41})\alpha_{12}) > 0.$$

In fact, if $\alpha_{43} > 0$, then $D > 0$ because of $\alpha_{43}\alpha_1(\alpha_{32} + \alpha_{42}) > 0$. If $\alpha_{43} = 0$, then $\alpha_{41} > 0$ and $\alpha_{13} > 0$, which implies that $D > 0$. Since we may exclude the cases of Propositions 3.11 and 3.12, renumbering a_1, a_2, a_3 and a_4 it suffices to consider the case where $\alpha_1 a_1 = \alpha_{12}a_2 + \alpha_{14}a_4$, which implies that $\alpha_3 = \alpha_{23} + \alpha_{43}$. Moreover, we have

$$"\alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3 \text{ or } \alpha_{23}a_3 + \alpha_{24}a_4" \text{ and } "\alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{43}a_3 \text{ or } \alpha_{42}a_2 + \alpha_{43}a_3".$$

Case 1). $\alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3$ and $\alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{43}a_3$. Then $\alpha_3 a_3 = \alpha_{32}a_2 + \alpha_{34}a_4$. This case is reduced to Proposition 3.13.

Case 2). $\alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3$ and $\alpha_4 a_4 = \alpha_{42}a_2 + \alpha_{43}a_3$. Then $\alpha_3 a_3 = \alpha_{31}a_1 + \alpha_{34}a_4$. Hence we get $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{42}$, $\alpha_3 = \alpha_{23} + \alpha_{43}$ and $\alpha_4 = \alpha_{14} + \alpha_{34}$. We introduce the order on the set I as follows: $(i,j) \le (i',j')$ if "$j < j'$" or "$j = j'$, $i \le i'$". Then we get the associated subsemigroup

$$S = < b_1 = e_1, \dots, b_5 = e_5, b_6, b_7, b_8 >$$

of $\mathbb{Z}^5$ through the method in Definition 3.6 where $b_6 = e_1 + e_2 - e_3$, $b_7 = e_1 + e_4 - e_3$ and $b_8 = e_2 + e_5 - e_4$. Take $\sum_{i=1}^{8} \lambda_i b_i \in \mathbb{Z}^5 = (\mu_1, \dots, \mu_5)$, with $0 \le \lambda_i < 1$. Then $\mu_1 = \lambda_1 + \lambda_6 + \lambda_7$, $\mu_2 = \lambda_2 + \lambda_6 + \lambda_8$, $\mu_3 = \lambda_3 - \lambda_6 - \lambda_7$, $\mu_4 = \lambda_4 + \lambda_7 - \lambda_8$ and $\mu_5 = \lambda_5 + \lambda_8$. If $\mu_3 = \lambda_3 - \lambda_6 - \lambda_7 = -1$, then we must have $\lambda_6 > 0$ and $\lambda_7 > 0$, which implies that $\mu_1 \ge 1$ and $\mu_2 \ge 1$. Thus, we may assume that $(1,1,-1,0,0) = b_6 \in S$. Hence S is saturated.

Case 3). $\alpha_2 a_2 = \alpha_{23}a_3 + \alpha_{24}a_4$ and $\alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{43}a_3$. We introduce the order on the set I as follows: $(i,j) \le (i',j')$ if "$j < j'$" or "$j = j'$, $i \le i'$". Then we get the associated subsemigroup

$$S = < b_1 = e_1, \dots, b_5 = e_5, b_6, b_7, b_8 >$$

of $\mathbb{Z}^5$ through the method in Definition 3.6 where $b_6 = e_1 + e_2 - e_3$, $b_7 = e_3 + e_4 - e_5$ and $b_8 = e_1 + e_4 - e_5$ In the similar way to Case 2) we can show that S is saturated.

Case 4). $\alpha_2 a_2 = \alpha_{23}a_3 + \alpha_{24}a_4$ and $\alpha_4 a_4 = \alpha_{42}a_2 + \alpha_{43}a_3$. In this case at most one α_{i1} appears. This contradicts the neatness of H. □

§4. Numerical semigroups of toric type.

In this section we consider a numerical semigroup H of quasi-toric type such that the associated saturated semigroup S determines the structure of the semigroup H in some sense. The section is concerned with the following neat numerical semigroups.

Definition 4.1. Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ such that it has a neat system of relations

$$\begin{cases} \alpha_1 a_1 = \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n a_n = \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1} \end{cases}$$

It is said to be *1-neat* if

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} = a_n.$$

By Proposition 2.6 we get the following example:

Example 4.2. Any non-symmetric numerical semigroup H with $M(H) = \{a_1, a_2, a_3\}$ is 1-neat.

When H is a 1-neat numercial semigroup using the following proposition we can give a basis for its relation $\mathbb{Z}-$module defined in Definition 3.3:

Proposition 4.3. *Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. Let $\mathbf{r}_i = (r_{i1}, r_{i2}, \dots, r_{in})$ be an element of the relation module R for H for each $i = 1, \dots, n-1$. Assume that*

$$\begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n-1} \\ r_{21} & r_{22} & \cdots & r_{2n-1} \\ \vdots & \vdots & & \vdots \\ r_{n-11} & r_{n-12} & \cdots & r_{n-1n-1} \end{vmatrix} = \pm a_n.$$

Then $\mathbf{r}_1, \dots, \mathbf{r}_{n-1}$ form a basis for the $\mathbb{Z}$-module R.

Proof. Let $\mathbf{s} = (s_1, \dots, s_n) \in R$. By the assumption and Lemma 3.4 we have an expression

$$\mathbf{s} = c_1\mathbf{r}_1 + \cdots + c_{n-1}\mathbf{r}_{n-1},\ c_i \in \mathbb{Q} \text{ for all } i.$$

We want to show that $c_i \in \mathbb{Z}$ for all i. It follows from the above equations that

$$c_i = \frac{1}{\pm a_n} \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n-1} \\ \vdots & \vdots & & \vdots \\ s_1 & s_2 & \cdots & s_{n-1} \\ \vdots & \vdots & & \vdots \\ r_{n-11} & r_{n-12} & \cdots & r_{n-1n-1} \end{vmatrix}$$

where $(s_1 s_2 \cdots s_{n-1})$ is the i-th row. Consider the following system of linear equations

$$\begin{cases} r_{11}a_1 + r_{12}a_2 + \cdots + r_{1n-1}a_{n-1} = -r_{1n}a_n. \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_1a_1 + s_2a_2 + \cdots + s_{n-1}a_{n-1} = -s_na_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ r_{n-11}a_1 + r_{n-12}a_2 + \cdots + r_{n-1n-1}a_{n-1} = -r_{n-1n}a_n \end{cases}$$

If $c_i \neq 0$, a_n divides

$$a_j \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n-1}. \\ \vdots & \vdots & & \vdots \\ s_1 & s_2 & \cdots & s_{n-1} \\ \vdots & \vdots & & \vdots \\ r_{n-11} & r_{n-12} & \cdots & r_{n-1n-1} \end{vmatrix}$$

for each $j = 1, \dots, n-1$. Since $(a_1, \dots, a_n) = 1$, we get

$$\begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n-1}. \\ \vdots & \vdots & & \vdots \\ s_1 & s_2 & \cdots & s_{n-1} \\ \vdots & \vdots & & \vdots \\ r_{n-11} & r_{n-12} & \cdots & r_{n-1n-1} \end{vmatrix} = \nu_i a_n$$

for some $\nu_i \in \mathbb{Z}$. Thus, c_i must be an integer. □

Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$ with a fixed neat system of relations. We set $N = \{(i,j) | \alpha_{ij} \neq 0\} - (n-1)$. We get the associated subsemigroup $S = < b_1, \dots, b_{N+n-1} >$ of $\mathbb{Z}^N$. Let k be a field. Let $\varphi_H : k[X] = k[X_1, \dots, X_n] \longrightarrow k[H] = k[t^h]_{h \in H}$ be a k-algebra homomorphism sending X_i to t^{a_i}, $\pi : k[Y] = k[Y_1, \dots, Y_{N+n-1}] \longrightarrow k[S] = k[T^b]_{b \in S}$ a k-algebra homomorphism sending Y_l to T^{b_l}, $\eta : k[Y] \longrightarrow k[X]$ a k-algebra homomorphism sending Y_l to $g_l =$

$X_j^{\alpha_{ij}}$ if b_l corresponds to $\alpha_{ij}a_j$ and $\zeta : k[\mathbb{N}^N] = k[t_1, \dots, k_N] \longrightarrow k[H]$ a k−algebra homomorphism sending t_i to $t^{w(g_i)}$ where the weight w on $k[X]$ is defined by $w(X_i) = a_i$ and $w(c) = 0$ for $c \in k^\times$. By the definition of b_l's ζ extends to $\zeta' : k[S] \longrightarrow k[H]$. Then we get $\varphi_H \circ \eta = \zeta' \circ \pi$, which implies that Ker $\varphi_H \supseteq \eta(\text{Ker } \pi)$.

Definition 4.4. Let the notation be as in the above. A neat numerical semigroup H is said to be *of toric type* if it is of quasi-toric type and we have an isomorphism $k[H] \cong k[S] \otimes_{k[Y]} k[X]$, that is to say, Ker $\varphi_H = \eta(\text{Ker } \pi)$.

Remark 4.5. A numerical semigroup of toric type is Weierstrass[3], where a numerical semigroup H is said to be *Weierstrass* if there is a pointed non-singular complete curve (C, P) over an algebraically closed field such that

$$H = \{n \in \mathbb{N} | \text{ there is a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

Example 4.6. i) Any non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$ is of toric type, because we know that the ideal Ker φ_H is generated by $X_1^{\alpha_1} - X_2^{\alpha_{12}}X_3^{\alpha_{13}}$, $X_2^{\alpha_2} - X_1^{\alpha_{21}}X_3^{\alpha_{23}}$ and $X_3^{\alpha_3} - X_1^{\alpha_{31}}X_2^{\alpha_{32}}$ (See 1)).
ii) For any integer $n \geq 5$ let H_n be a numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n+1, a_3 = 2n+3, a_4 = 2n+4, \dots, a_{n-1} = 2n+n-1\}$$

with the neat system of relations as in Example 3.2 iv). Then the ideal Ker φ_{H_n} is generated by

$$X_2^3 - X_1X_3,\ X_2X_j - X_1X_{j+1}(3 \leq j \leq n-2),\ X_2X_{n-1} - X_1^4,\ X_3X_j - X_2^2X_{j+1}(3 \leq j \leq n-2),$$

$$X_3X_{n-1} - X_2^2X_1^3,\ X_iX_j - X_{i-1}X_{j+1}(4 \leq i \leq n-2, i \leq j \leq n-2),\ X_iX_{n-1} - X_{i-1}X_1^3(4 \leq i \leq n-1)$$ [2].

It is proved that H_n is of toric type.

As we see in the above examples, to show that a numerical semigroup H of quasi-toric type is of toric type we need to determine a set of generators for the ideal Ker φ_H. In the case where H is a neat numerical semigroup generated by four elements a set of generators for the ideal Ker φ_H is determined [3]. Hence considering Proposition 4.3 we get the following theorem:

Theorem 4.7. *A 1-neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$ is of toric type.*

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