

# On 6-semigroups generated by 4 elements from which affine toric varieties can be constructed

Jiryo KOMEDA

Department of Liberal Arts

## Abstract

This paper is devoted to constructing an affine toric variety from a 6-semigroup generated by 4 elements such that the monomial curve associated with the 6-semigroup is a specialization of the toric variety. A *6-semigroup* means a numerical semigroup starting with 6. We know that if we can construct such an affine toric variety from a given numerical semigroup, then the numerical semigroup is Weierstrass<sup>1)</sup>. For that reason we try to construct an affine toric variety from a 6-semigroup generated by 4 elements which is not known to be Weierstrass.

Key Words: Affine toric variety, 6-semigroup, Weierstrass semigroup

## §1. Introduction.

Let  $\mathbb{N}$  be the additive semigroup of non-negative integers. A subsemigroup  $H$  of  $\mathbb{N}$  is called a *numerical semigroup* if its complement  $\mathbb{N} \setminus H$  is a finite set. Let  $C$  be a complete non-singular irreducible algebraic curve over an algebraically closed field  $k$  of characteristic 0. For any point  $P$  of  $C$ ,  $H(P)$  denotes the set of integers which are pole orders at  $P$  of regular functions on  $C \setminus \{P\}$ . Then  $H(P)$  is a numerical semigroup. A numerical semigroup  $H$  is said to be *Weierstrass* if there exists a pair  $(C, P)$  such that  $H = H(P)$ . For a positive integer  $n$  a numerical semigroup  $H$  is said to be an  *$n$ -semigroup* if the minimum positive integer in  $H$  is  $n$ . It is known that any  $n$ -semigroup is Weierstrass if  $n \leq 5$ <sup>2),1),3)</sup>. Moreover, all numerical semigroups generated by two or three elements are Weierstrass<sup>4)</sup>. We devote ourselves to the study of 6-semigroups generated by 4 elements. First, we list up the semigroups among them which are not known to be Weierstrass. In Section 2 we investigate semigroups from which we can succeed in constructing affine toric varieties, in some sense the monomial curve associated with the semigroup is a specialization of the affine toric variety. There are 6-semigroups generated by 4 elements from which we fail to construct affine toric varieties. In Section 3 we treat such 6-semigroups and explain why we do not succeed in getting affine toric varieties from them.

## §2. 6-semigroups from which we can construct affine toric varieties.

Let  $H$  be a 6-semigroup generated by 4 elements with the minimal set  $M(H) = \{a_0 = 6, a_1, a_2, a_3\}$  of generators. We can show the following whose proof will be given in §5).

**Remark 2.1.** If  $M(H)$  contains an element whose residue is 2 (resp. 4) modulo 6, then  $H$  is Weierstrass.

We must investigate 6-semigroups  $H$  with  $M(H) = \{a_0 = 6, a_1, a_2, a_3\}$  satisfying one of the following:

- i)  $a_1 \equiv 1 \pmod{6}$ ,  $a_2 \equiv 2 \pmod{6}$ ,  $a_3 \equiv 3 \pmod{6}$ , ii)  $a_1 \equiv 1 \pmod{6}$ ,  $a_2 \equiv 2 \pmod{6}$ ,  $a_3 \equiv 5 \pmod{6}$ ,
- iii)  $a_1 \equiv 1 \pmod{6}$ ,  $a_2 \equiv 3 \pmod{6}$ ,  $a_3 \equiv 4 \pmod{6}$ , iv)  $a_1 \equiv 1 \pmod{6}$ ,  $a_2 \equiv 3 \pmod{6}$ ,  $a_3 \equiv 5 \pmod{6}$ ,
- v)  $a_1 \equiv 1 \pmod{6}$ ,  $a_2 \equiv 4 \pmod{6}$ ,  $a_3 \equiv 5 \pmod{6}$ , vi)  $a_1 \equiv 2 \pmod{6}$ ,  $a_2 \equiv 3 \pmod{6}$ ,  $a_3 \equiv 5 \pmod{6}$ ,
- vii)  $a_1 \equiv 3 \pmod{6}$ ,  $a_2 \equiv 4 \pmod{6}$ ,  $a_3 \equiv 5 \pmod{6}$ .

Except some cases of iii) (resp. vi)) we can get affine toric varieties. In fact, we get the following theorem:

**Theorem 2.2.** *If a 6-semigroup  $H$  satisfies one of the conditions i), ii), iv), v) and vii), then we can construct an affine toric variety from it, hence it is Weierstrass.*

*Proof.* The way to get an affine toric variety from a numerical semigroup is the following: First, we determine a *minimal system of relations* among  $a_0 = 6, a_1, a_2$  and  $a_3$ , which means that the ideal of the monomial curve attached to  $H$  is generated by the polynomials corresponding to the relations. Secondly, we find out a fundamental system of relations from the minimal system of relations. A *fundamental system of relations* means a minimal set of generators for the free module of relations among  $a_0 = 6, a_1, a_2$  and  $a_3$ . By Lemma 3.4 in [6] the rank of the free module is 3. Thirdly, we get the semigroup  $S$  of  $\mathbb{Z}^m$  for some  $m$  from the above fundamental system. Lastly, we prove that  $S$  is *saturated*, i.e., if  $r \in \mathbb{Z}^m$  satisfies  $nr \in S$  for some positive integer  $n$ , then  $r \in S$ . Then  $\text{Spec } k[S]$  is an affine toric variety which we try to find.

Here we give an explicit proof in the case of 6-semigroups  $H$  of type ii), i.e.,

$$M(H) = \{a_0 = 6, a_1 = 6m_1 + 1, a_2 = 6m_2 + 2, a_3 = 6m_5 + 5\},$$

because similar methods work well in the other cases. We divide such 6-semigroups into 4 cases.

- ii-1)  $m_1 + m_5 \leq 3m_2$ ,  $m_1 + m_2 \leq 3m_5 + 2$ .
- ii-2)  $m_1 + m_5 \leq 3m_2$ ,  $m_1 + m_2 > 3m_5 + 2$ ,  $3m_5 + m_1 + 2 \leq 2m_2$ .
- ii-3)  $m_1 + m_5 \leq 3m_2$ ,  $m_1 + m_2 > 3m_5 + 2$ ,  $3m_5 + m_1 + 2 > 2m_2$ .
- ii-4)  $m_1 + m_5 > 3m_2$ ,  $m_1 + m_2 \leq 3m_5 + 2$ .

We note that the case where  $m_1 + m_5 \geq 3m_2$  and  $m_1 + m_2 \geq 3m_5 + 2$  does not occur, because these two inequalities imply that  $m_1 \geq m_2 + m_5 + 1$ , which is a contradiction.

ii-1) We have a subset of a minimal system of relations:

$$(m_1 + m_5 + 1)a_0 = a_1 + a_3, \quad 2a_1 = (2m_1 - m_2)a_0 + a_2,$$

$$2a_2 = (2m_2 - 2m_5 - 1)a_0 + 2a_3 \quad (\text{resp. } 3a_2 = (3m_2 - m_1 - m_5)a_0 + a_1 + a_3),$$

$$3a_3 = (3m_5 - m_1 - m_2 + 2)a_0 + a_1 + a_2 \quad (\text{resp. } 2a_3 = (2m_5 - 2m_2 + 1)a_0 + 2a_2)$$

if  $m_2 > m_5$  (resp.  $m_2 \leq m_5$ ). Hence,  $H$  is 1-neat (See Definition 4.10 in [7]). Therefore, we get an affine toric variety (See Theorem 4.11 in [7]).

ii-2) We have a subset of a minimal system of relations:

$$(m_1 + m_5 + 1)a_0 = a_1 + a_3, \quad 2a_1 = (2m_1 - m_2)a_0 + a_2,$$

$$2a_2 = (2m_2 - 3m_5 - m_1 - 2)a_0 + a_1 + 3a_3, \quad 4a_3 = (4m_5 - m_2 + 3)a_0 + a_2,$$

which imply that  $H$  is 1-neat. Hence, we can construct an affine toric variety.

ii-3) We have a minimal system of relations:

$$(m_1 + m_5 + 1)a_0 = a_1 + a_3, \tag{1}$$

$$2a_1 = (2m_1 - m_2)a_0 + a_2, \tag{2}$$

$$2a_2 = (2m_2 - 2m_5 - 1)a_0 + 2a_3, \quad (3)$$

$$4a_3 = (4m_5 - m_2 + 3)a_0 + a_2, \quad (4)$$

$$(m_1 + m_2 - 3m_5 - 2)a_0 + 3a_3 = a_1 + a_2, \quad (5)$$

$$(m_5 + m_2 - m_1 + 1)a_0 + a_1 = a_2 + a_3. \quad (6)$$

In fact, using Lemma 4.12 in 1) we can show that the ideal  $I_H = \text{Ker } \varphi_H$ , where

$$\varphi_H : k[X_0, X_1, X_2, X_3] \longrightarrow k[H]$$

is the  $k$ -algebra homomorphism sending  $X_i$  to  $t^{a_i}$ , is generated by

$$X_0^{m_1+m_5+1} - X_1X_3, X_1^2 - X_0^{2m_1-m_2}X_2, X_2^2 - X_0^{2m_2-2m_5-1}X_3^2, \\ X_3^4 - X_0^{4m_5-m_2+3}X_2, X_0^{m_1+m_2-3m_5-2}X_3^3 - X_1X_2 \text{ and } X_0^{m_5+m_2-m_1+1}X_1 - X_2X_3.$$

(1), (5) and (6) form a fundamental system of relations. In fact, we get

$${}^t(1) + (6) = (2), {}^t(5) + {}^t(6) = (3) \text{ and } {}^t(1) + (5) = (4)$$

where  ${}^t(n)$  means the equation whose right side (resp. left side) is the left side (resp. right side) of the equation (n). The relations (1), (5) and (6) have expressions

$$(\alpha'_0 + \alpha''_0 + \alpha'''_0)a_0 = \alpha'_1a_1 + \alpha_{03}a_3,$$

$$\alpha'''_0a_0 + (\alpha_{03} + \alpha_{23})a_3 = \alpha_{12}a_2 + \alpha'_1a_1,$$

$$\alpha''_0a_0 + \alpha'_1a_1 = \alpha_{32}a_2 + \alpha_{03}a_3$$

respectively, where

$$\alpha'_0 = m_1 + 3m_5 - 2m_2 + 2, \alpha''_0 = m_5 + m_2 - m_1 + 1, \alpha'''_0 = m_1 + m_2 - 3m_5 - 2, \alpha_{03} = 1, \alpha_{23} = 2,$$

$$\alpha_{12} = 1, \alpha_{32} = 1 \text{ and } \alpha'_1 = 1.$$

We attach the vectors  $b_1 = e_1, b_2 = e_2, b_3 = e_3, b_4 = e_4$  and  $b_5 = e_5 \in \mathbb{Z}^5$  to  $\alpha'_0a_0, \alpha''_0a_0, \alpha'''_0a_0, \alpha'_1a_1$  and  $\alpha_{23}a_3$  respectively, where  $e_i$  denotes the vector whose  $i$ -th component is 1 and whose  $j$ -th component is 0 if  $j \neq i$ . Hence, the vector  $b_6 = (1, 1, 1, -1, 0)$  is attached to  $\alpha_{03}a_3$ . Moreover, we attach  $b_7 = (-1, -1, 0, 2, 1)$  and  $b_8 = (1, 2, 1, -2, 0)$  to  $\alpha_{12}a_2$  and  $\alpha_{32}a_2$  respectively. Let  $S$  be the subsemigroup of  $\mathbb{Z}^5$  generated by  $b_1, \dots, b_8$ . To check that  $S$  is saturated it suffices to show that  $\sum_{i=1}^8 \mathbb{R}_+ b_i \cap \mathbb{Z}^5 \subseteq S$ ,

where  $\mathbb{R}_+$  denotes the set of non-negative real numbers. Let  $b \in \sum_{i=1}^8 \mathbb{R}_+ b_i \cap \mathbb{Z}^5$ . We may assume that

$$b = \sum_{i=1}^8 \lambda_i b_i \in \mathbb{Z}^5 \text{ with } 0 \leq \lambda_i < 1, \text{ all } i. \text{ Then we get}$$

$$b = (\lambda_1 + \lambda_6 - \lambda_7 + \lambda_8, \lambda_2 + \lambda_6 - \lambda_7 + 2\lambda_8, \lambda_3 + \lambda_6 + \lambda_8, \lambda_4 - \lambda_6 + 2\lambda_7 - 2\lambda_8, \lambda_5 + \lambda_7).$$

Since  $\lambda_4 - \lambda_6 + 2\lambda_7 - 2\lambda_8 \geq -2$ , it suffices to show that if  $\lambda_4 - \lambda_6 + 2\lambda_7 - 2\lambda_8 = -2$  or  $-1$ , then  $b \in S$ . Let  $\lambda_4 - \lambda_6 + 2\lambda_7 - 2\lambda_8 = -2$ . Then

$$\lambda_1 + \lambda_6 - \lambda_7 + \lambda_8 \geq 2, \lambda_2 + \lambda_6 - \lambda_7 + 2\lambda_8 \geq 2 \text{ and } \lambda_3 + \lambda_6 + \lambda_8 \geq 2.$$

Hence, we may assume that  $b = (2, 2, 2, -2, 0) = 2b_6$ , which implies that  $b \in S$ . If  $\lambda_4 - \lambda_6 + 2\lambda_7 - 2\lambda_8 = -1$ , we get  $b \in S$  in a way similar to the above. Therefore, we get an affine toric variety  $\text{Spec } k[S]$ .

ii-4) We have a minimal system of relations:

$$(3m_2 + 1)a_0 = 3a_2, \quad (7)$$

$$2a_1 = (2m_1 - m_2)a_0 + a_2, \quad (8)$$

$$2a_3 = (2m_5 - 2m_2 + 2)a_0 + 2a_2, \quad (9)$$

$$(m_1 + m_5 - 3m_2)a_0 + 3a_2 = a_1 + a_3, \quad (10)$$

$$(m_2 + m_5 - m_1 + 1)a_0 + a_1 = a_2 + a_3, \quad (11)$$

$$(m_1 + 2m_2 - m_5)a_0 + a_3 = a_1 + 2a_2. \quad (12)$$

Let  $J$  be the ideal generated by

$$X_0^{3m_2+1} - X_2^3, X_1^2 - X_0^{2m_1-m_2}X_2, X_3^2 - X_0^{2m_5-2m_2+1}X_2^2, \\ X_0^{m_1+m_5-3m_2}X_3^2 - X_1X_3, X_0^{m_2+m_5-m_1+1}X_1 - X_2X_3 \text{ and } X_0^{m_1+2m_2-m_5}X_3 - X_1X_2^2.$$

We need to show that  $I_H = J$ . It suffices to show that  $I_H \subseteq J$ . We may take as generators for the ideal  $I_H$  one of the following types<sup>8)</sup>:

(I)  $F = X_i^{\nu_i} - X_j^{\mu_j} X_k^{\mu_k} X_l^{\mu_l}$ , where  $i, j, k$  and  $l$  are distinct, and  $\nu_i > 0$ ,  $\mu_j > 0$ ,  $\mu_k \geq 0$ ,  $\mu_l \geq 0$ ,

(II)  $F = X_i^{\nu_i} X_j^{\nu_j} - X_k^{\mu_k} X_l^{\mu_l}$ , where  $i, j, k$  and  $l$  are distinct, and  $\nu_i > 0$ ,  $\nu_j > 0$ ,  $\mu_k > 0$ ,  $\mu_l > 0$ .

First, we define the weight of  $X_i$  by  $a_i$ . Moreover, we set

$$\alpha_i = \min\{\alpha > 0 \mid \alpha a_i \in \text{the semigroup generated by } a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_3\}, \text{ all } i.$$

Let  $F \in I_H$  be of type (I). If  $\nu_i > \alpha_i$ , then we may decrease the weighted degree of  $F$  or reduce this case to the case (II). Let  $\nu_0 = \alpha_0$ . Then we may assume that  $F = X_2^3 - X_1^{\mu_1} X_3^{\mu_3}$ , because  $X_0^{\alpha_0} - X_2^3 \in J$ . Moreover, since  $X_1^2 - X_0^{2m_1-m_2}X_2$  and  $X_3^2 - X_0^{2m_5-2m_2+1}X_2^2$ , we may assume that  $F = X_2^3 - X_1X_3$ , which contradicts  $m_1 + m_3 > 3m_2$ . In the case where  $1 \leq i \leq 3$  a method similar to the case  $i = 0$  works well. Let  $F \in I_H$  be of type (II). We may assume that  $i = 0$ . If  $(k, l) = (2, 3)$ , using  $X_0^{m_2+m_5-m_1+1}X_1 - X_2X_3 \in J$  we may decrease the weighted degree of  $F$ . If  $(k, l) = (1, 3)$ , the same method as in  $(k, l) = (2, 3)$  works well. Let  $(k, l) = (1, 2)$ . If  $\mu_1 \geq 2$ , then use  $X_1^2 - X_0^{2m_1-m_2}X_2 \in J$ . If  $\mu_1 = 1$  and  $\mu_2 \geq 2$ , we can use  $X_0^{m_1+2m_2-m_5}X_3 - X_1X_2^2 \in J$ . Hence, we may assume that  $F = X_0^{\nu_0}X_3^{\nu_3} - X_1X_2$ . If  $\nu_3 \geq 2$ , using  $X_3^2 - X_0^{2m_5-2m_2+1}X_2^2 \in J$  we may decrease the weighted degree of  $F$ . The remaining case is  $\nu_3 = 1$ . In this case we have  $\nu_0a_0 + a_3 = a_1 + a_2 \equiv 3 \pmod{6}$ , which is a contradiction.

By the way (10), (11) and (12) forms a fundamental system of relations. In fact, we get

$$(11) + (12) = (7), {}^t(10) + {}^t(12) = (8) \text{ and } {}^t(10) + {}^t(11) = (9).$$

The relations (10), (11) and (12) have expressions

$$\alpha'_0 a_0 + (\alpha'_2 + \alpha''_2) a_2 = \alpha'_1 a_1 + \alpha'_3 a_3,$$

$$\alpha''_0 a_0 + \alpha'_1 a_1 = \alpha'_2 a_2 + \alpha'_3 a_3,$$

$$\alpha'''_0 a_0 + \alpha'_3 a_3 = \alpha'_1 a_1 + \alpha''_2 a_2$$

respectively, where

$$\alpha'_0 = m_1 + m_5 - 3m_2, \alpha''_0 = m_2 + m_5 - m_1 + 1, \alpha'''_0 = m_1 + 2m_2 - m_5, \alpha'_1 = 1, \alpha'_2 = 1, \alpha''_2 = 2 \text{ and } \alpha'_3 = 1.$$

We attach the vectors  $b_1 = e_1, b_2 = e_2, b_3 = e_3$  and  $b_4 = e_4 \in \mathbb{Z}^4$  to  $\alpha'_0 a_0, \alpha'_2 a_2, \alpha''_2 a_2$  and  $\alpha'_1 a_1$  respectively. Hence, the vector  $b_5 = (1, 1, 1, -1)$  is attached to  $\alpha'_3 a_3$ . Moreover, we attach  $b_6 =$

$(1, 2, 1, -2)$  and  $b_7 = (-1, -1, 0, 2)$  to  $\alpha_0'' a_0$  and  $\alpha_0''' a_0$  respectively. Let  $b = \sum_{i=1}^7 \lambda_i b_i \in \mathbb{Z}^4$  with  $0 \leq \lambda_i < 1$ , all  $i$ . Then we obtain

$$b = (\lambda_1 + \lambda_5 + \lambda_6 - \lambda_7, \lambda_2 + \lambda_5 + 2\lambda_6 - \lambda_7, \lambda_3 + \lambda_5 + \lambda_6, \lambda_4 - \lambda_5 - 2\lambda_6 + 2\lambda_7).$$

If  $\lambda_4 - \lambda_5 - 2\lambda_6 + 2\lambda_7 = -2$ , then we may assume that  $b = (2, 2, 2, -2) = 2b_5 \in S$ . If  $\lambda_4 - \lambda_5 - 2\lambda_6 + 2\lambda_7 = -1$ , then we may assume that  $b = (1, 1, 1, -1) = b_5 \in S$ . Hence, the semigroup  $S$  is saturated, which implies that  $\text{Spec } k[S]$  is our desired affine toric variety.

In the case of i) or iv) or v) or vii) we can also get an affine toric variety from the 6-semigroup  $H$  as follows:

i) In a way similar to ii-4) we obtain an affine toric variety.

iv) We divide the 6-semigroups into 4 cases.

iv-1)  $2m_3 \geq m_1 + m_5$ ,  $2m_1 \geq m_3 + m_5 + 1$ . In this case the semigroups are 1-neat.

iv-2)  $2m_3 \geq m_1 + m_5$ ,  $2m_1 < m_3 + m_5 + 1$ .  $2m_5 + 1 \geq m_1 + m_3$ . In this case the semigroups are also 1-neat.

iv-3)  $2m_3 \geq m_1 + m_5$ ,  $2m_1 < m_3 + m_5 + 1$ .  $2m_5 + 1 < m_1 + m_3$ . In a way similar to ii-3) we get an affine toric variety.

iv-4)  $2m_3 < m_1 + m_5$ . By the way like ii-4) we can construct an affine toric variety.

v) We divide the semigroups into 5 cases.

v-1)  $m_1 + m_5 \leq 3m_4 + 1$ ,  $m_1 > m_4$ . The semigroups of this type are 1-neat.

v-2)  $m_1 + m_5 \leq 3m_4 + 1$ ,  $m_1 \leq m_4$ ,  $m_4 + m_5 + 1 \leq 3m_1$ . These 5-semigroups are also 1-neat.

v-3)  $m_1 + m_5 \leq 3m_4 + 1$ ,  $m_1 \leq m_4$ ,  $m_4 + m_5 + 1 > 3m_1$ ,  $3m_1 + m_5 \leq 2m_4$ . In this case we can show that the semigroups are 1-neat.

v-4)  $m_1 + m_5 \leq 3m_4 + 1$ ,  $m_1 \leq m_4$ ,  $m_4 + m_5 + 1 > 3m_1$ ,  $3m_1 + m_5 > 2m_4$ . In a way similar to ii-3) we get an affine toric variety.

v-5)  $m_1 + m_5 > 3m_4 + 1$ . In a way similar to ii-4) we obtain an affine toric variety.

vii) In a way similar to ii-4) we can construct an affine toric variety.

The details will be given in 5). □

### §3. 6-semigroups from which we can not construct affine toric varieties.

Let  $H$  be a 6-semigroup generated by 4 elements with the minimal set

$$M(H) = \{a_0 = 6, a_1 = 6m_1 + 1, a_2 = 6m_3 + 3, a_3 = 6m_4 + 4\}$$

of generators.

Case of  $m_1 \leq m_4$  and  $2m_1 + m_4 \leq 2m_3$ . We have a minimal system of relations:

$$(2m_1 + m_4 + 1)a_0 = 2a_1 + a_3, \quad (13)$$

$$3a_1 = (3m_1 - m_3)a_0 + a_2, \quad (14)$$

$$2a_2 = (2m_3 - 2m_1 - m_4)a_0 + 2a_1 + a_3, \quad (15)$$

$$2a_3 = (2m_4 - 2m_1 + 1)a_0 + 2a_1, \quad (16)$$

$$(m_3 + m_4 - m_1 + 1)a_0 + a_1 = a_2 + a_3, \quad (17)$$

$$(m_1 + m_3 - m_4)a_0 + a_3 = a_1 + a_2. \quad (18)$$

In fact, using Lemma 4.12 in 1) we can show that the ideal  $I_H$  is generated by

$$X_0^{2m_1+m_4+1} - X_1^2 X_3, X_1^3 - X_0^{3m_1-m_3} X_2, X_2^2 - X_0^{2m_3-2m_1-m_4} X_1^2 X_3,$$

$$X_3^2 - X_0^{2m_4-2m_1+1} X_1^2, X_0^{m_3+m_4-m_1+1} X_1 - X_2 X_3 \text{ and } X_0^{m_1+m_3-m_4} X_3 - X_1 X_2.$$

Moreover, (14), (17) and (18) form a fundamental system of relations. In practice, we get

$${}^t(14) + {}^t(17) = (13), {}^t(14) + {}^t(18) = (15) \text{ and } {}^t(17) + {}^t(18) = (16).$$

The relations (14), (18) and (17) have expressions

$$(\alpha'_1 + \alpha''_1)a_1 = \alpha'_0 a_0 + \alpha'_2 a_2,$$

$$(\alpha'_0 + \alpha''_0)a_0 + \alpha_{23}a_3 = \alpha'_1 a_1 + \alpha'_2 a_2,$$

$$(\alpha'_0 + \alpha''_0 + \alpha'''_0)a_0 + \alpha'_1 a_1 = \alpha'_2 a_2 + \alpha_{03}a_3$$

respectively, where

$$\alpha'_0 = 3m_1 - m_3, \alpha''_0 = 2m_3 - 2m_1 - m_4, \alpha'''_0 = 2m_4 - 2m_1 + 1,$$

$$\alpha'_1 = 1, \alpha''_1 = 2, \alpha'_2 = 1, \alpha_{03} = 1 \text{ and } \alpha_{23} = 1.$$

We attach the vectors  $b_1 = e_1, b_2 = e_2, b_3 = e_3, b_4 = e_4$  and  $b_5 = e_5 \in \mathbb{Z}^5$  to  $\alpha'_1 a_1, \alpha''_1 a_1, \alpha'_0 a_0, \alpha''_0 a_0$  and  $\alpha'''_0 a_0$  respectively. Hence, the vector  $b_6 = (1, 1, -1, 0, 0)$  is attached to  $\alpha'_2 a_2$ . Moreover, we attach  $b_7 = (2, 1, -2, -1, 0)$  and  $b_8 = (0, -1, 2, 1, 1)$  to  $\alpha_{23}a_3$  and  $\alpha_{03}a_3$  respectively. Let  $b = \sum_{i=1}^8 \lambda_i b_i \in \mathbb{Z}^5$  with  $0 \leq \lambda_i < 1$ , all  $i$ . Then we obtain

$$b = (\lambda_1 + \lambda_6 + 2\lambda_7, \lambda_2 + \lambda_6 + \lambda_7 - \lambda_8, \lambda_3 - \lambda_6 - 2\lambda_7 + 2\lambda_8, \lambda_4 - \lambda_7 + \lambda_8, \lambda_5 + \lambda_8).$$

We get  $\lambda_3 - \lambda_6 - 2\lambda_7 + 2\lambda_8 \geq -2$ . It suffices to show that if  $\lambda_3 - \lambda_6 - 2\lambda_7 + 2\lambda_8 = -2$  or  $-1$ , then  $b$  belongs to the semigroup  $S$  generated by  $b_1, \dots, b_8$ . Let  $\lambda_3 - \lambda_6 - 2\lambda_7 + 2\lambda_8 = -2$ . Then

$$\lambda_1 + \lambda_6 + 2\lambda_7 \geq 2 \text{ and } \lambda_2 + \lambda_6 + \lambda_7 - \lambda_8 \geq 2.$$

Hence, we may assume that  $b = (2, 2, -2, 0, 0) = 2b_6$ , which implies that  $b \in S$ . If  $\lambda_3 - \lambda_6 - 2\lambda_7 + 2\lambda_8 = -1$ , we get  $b \in S$  in a way similar to the above. Hence, the semigroup  $S$  is saturated. Thus, we get an affine toric variety  $\text{Spec } k[S]$  from the 6-semigroup  $H$ .

Case of  $m_1 \leq m_4$  and  $2m_1 + m_4 > 2m_3$ . We fail to get an affine toric variety from it. From now on, we explain its details. First, we can show that the following relations form a minimal system of relations:

$$(2m_3 + 1)a_0 = 2a_2, \tag{19}$$

$$3a_1 = (3m_1 - m_3)a_0 + a_2, \tag{20}$$

$$2a_3 = (2m_4 + 1 - 2m_1)a_0 + 2a_1, \tag{21}$$

$$(m_3 + m_4 - m_1 + 1)a_0 + a_1 = a_2 + a_3 \tag{22}$$

$$(2m_1 + m_4 - 2m_3)a_0 + 2a_2 = 2a_1 + a_3. \tag{23}$$

$$(m_1 + m_3 - m_4)a_0 + a_3 = a_1 + a_2, \tag{24}$$

In fact, it is not difficult to prove that the ideal  $I_H$  is generated by

$$X_0^{2m_3+1} - X_2^2, X_1^3 - X_0^{3m_1-m_3} X_2, X_3^2 - X_0^{2m_4-2m_1+1} X_1^2,$$

$$X_0^{m_3+m_4-m_1+1}X_1 - X_2X_3, X_0^{2m_1+m_4-2m_3}X_2^2 - X_1^2X_3 \text{ and } X_0^{m_1+m_3-m_4}X_3 - X_1X_2.$$

Moreover, (20), (22) and (24) form a fundamental system of relations. In effect, we get

$$(22) + (24) = (19), {}^t(22) + (24) = (21) \text{ and } {}^t(20) + {}^t(24) = (23).$$

The relations (20), (24) and (22) have expressions

$$(\alpha'_1 + \alpha''_1)a_1 = (\alpha'_0 + \alpha'''_0)a_0 + \alpha'_2a_2,$$

$$\alpha'_0a_0 + \alpha'_3a_3 = \alpha'_1a_1 + \alpha'_2a_2,$$

$$(\alpha'_0 + \alpha''_0)a_0 + \alpha'_1a_1 = \alpha'_2a_2 + \alpha'_3a_3$$

respectively, where

$$\alpha'_0 = m_1 + m_3 - m_4, \alpha''_0 = 2m_4 - 2m_1 + 1, \alpha'''_0 = 2m_1 + m_4 - 2m_3, \alpha'_1 = 1, \alpha''_1 = 2, \alpha'_2 = 1 \text{ and } \alpha'_3 = 1.$$

We attach the vectors  $b_1 = e_1, b_2 = e_2, b_3 = e_3$  and  $b_4 = e_4 \in \mathbb{Z}^4$  to  $\alpha'_1a_1, \alpha''_1a_1, \alpha'_0a_0$  and  $\alpha'''_0a_0$  respectively. Hence, the vector  $b_5 = (1, 1, -1, -1)$  is attached to  $\alpha'_2a_2$ . Moreover, we attach  $b_6 = (2, 1, -2, -1)$  and  $b_7 = (2, 2, -4, -2)$  to  $\alpha'_3a_3$  and  $\alpha''_0a_0$  respectively. In this case we get  $\frac{1}{2}b_7 = (1, 1, -2, -1) \in \sum_{i=1}^7 \mathbb{R}_+b_i \cap \mathbb{Z}^4$ , but  $\frac{1}{2}b_7 \notin S$  where  $S$  is the semigroup generated by  $b_1, \dots, b_7$ . Thus, the semigroup  $S$  is not saturated. It means that  $\text{Spec } k[S]$  is not an affine toric variety.  $\square$

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